Calculus of Variations and Optimal Control: Continuous Systems (various problem conditions)

- $x(0)$ & t_f given
- continuous time LQR
- t_f given, $x(0)$ & $x(t_f)$ [partly] given
- t_f unconstrained, $x(0)$, $x(t_f)$ [partly] given (example: min time)
- Integral constraint over path (Eg. fixed path length)
- Equality constraints over *functions* of the state/control

Continuous System
x(0) and t_f are known

- Given:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \quad \mathbf{x}(t_0)\ given \quad t \in [t_0, t_f]$$

- Find **u**(t) that minimizes

$$J = \phi\left(\mathbf{x}(t_f), t_f\right) + \int_{t_0}^{t_f} \mathcal{L}\left(\mathbf{x}(t), \mathbf{u}(t), t\right) dt$$

- Use Lagrange multiplier *functions*

$$J_A = \phi\left(\mathbf{x}(t_f), t_f\right) + \int_{t_0}^{t_f} \mathcal{L}\left(\mathbf{x}(t), \mathbf{u}(t), t\right) dt + \int_{t_0}^{t_f} \lambda^T(t)\left[\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) - \dot{\mathbf{x}}(t)\right] dt$$

$$J_A = \phi\left(\mathbf{x}(t_f), t_f\right) + \int_{t_0}^{t_f} \mathcal{L}\left(\mathbf{x}(t), \mathbf{u}(t), t\right) dt + \int_{t_0}^{t_f} \lambda^T(t)\left[\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) - \dot{\mathbf{x}}(t)\right] dt$$

- Define a Hamiltonian

$$\mathcal{H}\left(\mathbf{x}(t), \mathbf{u}(t), t\right) = \mathcal{L}\left(\mathbf{x}(t), \mathbf{u}(t), t\right) + \lambda^T(t)\mathbf{f}\left(\mathbf{x}(t), \mathbf{u}(t), t\right)$$

- Plug into Lagrangian and integrate by parts $\lambda^T(t)\dot{\mathbf{x}}(t)$

$$J = \phi\left(\mathbf{x}(t_f), t_f\right) - \lambda^T\left(t_f\right)\mathbf{x}\left(t_f\right) + \lambda^T\left(t_0\right)\mathbf{x}\left(t_0\right) + \int_{t_0}^{t_f} \left[\mathcal{H}\left(\mathbf{x}(t), \mathbf{u}(t), t\right) + \dot{\lambda}^T\mathbf{x}(t)\right] dt$$

- Its variation (w.r.t. changes in **x** and **u**) is:

$$\delta J = \left[\left(\frac{\partial\phi}{\partial\mathbf{x}} - \lambda^T\right)\delta\mathbf{x}\right]_{t=t_f} + \left[\lambda^T\delta\mathbf{x}\right]_{t=t_0} + \int_{t_0}^{t_f} \left[\left(\frac{\partial\mathcal{H}}{\partial\mathbf{x}} + \dot{\lambda}^T\right)\delta\mathbf{x} + \frac{\partial\mathcal{H}}{\partial\mathbf{u}}\delta\mathbf{u}\right] dt$$

$$\delta J = \left[\left(\frac{\partial \phi}{\partial \mathbf{x}} - \lambda^T \right) \delta \mathbf{x} \right]_{t=t_f} + [\lambda^T \delta \mathbf{x}]_{t=t_0} + \int_{t_0}^{t_f} \left[\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}} + \dot{\lambda}^T \right) \delta \mathbf{x} + \frac{\partial \mathcal{H}}{\partial \mathbf{u}} \delta \mathbf{u} \right] dt$$

- Zeroing it out (+ keeping dynamics requirements) leads to the Euler Lagrange equations:

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \\
\dot{\lambda}^T(t) &= -\frac{\partial \mathcal{L}}{\partial \mathbf{x}} - \lambda^T(t) \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \\
\lambda^T(t_f) &= \frac{\partial \phi}{\partial \mathbf{x}(t_f)} \\
\frac{\partial \mathcal{H}}{\partial \mathbf{u}} &= 0 \\
& \mathbf{x}_0 \ given
\end{aligned}$$

- If $\mathcal{L}$ and **f** are not explicit functions of t (happens often) then the Hamiltonian is constant (i.e. invariant) on the system's trajectory!

$$\begin{aligned}
\frac{d\mathcal{H}}{dt} &= \frac{d}{dt}\left[\mathcal{L}\left(\mathbf{x}(t),\mathbf{u}(t)\right)+\lambda^T(t)\mathbf{f}\left(\mathbf{x}(t),\mathbf{u}(t)\right)\right] \\
& \frac{\partial L}{\partial \mathbf{x}}\mathbf{f}+\frac{\partial L}{\partial \mathbf{u}}\dot{\mathbf{u}}+\dot{\lambda}^T\mathbf{f}+\lambda^T\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\underbrace{\dot{\mathbf{x}}}_{=\mathbf{f}}+\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\dot{\mathbf{u}}\right) \\
&= \frac{\partial \mathcal{H}}{\partial \mathbf{u}}\dot{\mathbf{u}}+\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}+\dot{\lambda}^T\right)\mathbf{f}
\end{aligned}$$

- On the extremal path both terms zero out, meaning that $H(t)$ = *constant*

Continuous Time LQR

- Given $\mathbf{x}(0)$, t_f and a linear system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

- Find a path that brings (some of) **x** close to zero without allowing it to vary to much on the way and without too much control energy. I.e. find **u** that minimizes

$$J = \frac{1}{2}\left(\mathbf{x}^T\mathbf{S}_f\mathbf{x}\right)_{t=t_f} + \frac{1}{2}\int_{t_0}^{t_f}\left(\mathbf{x}\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right)$$

  where **S,Q** and **R** are positive definite.

- Note: this can easily be generalized to *time varying* systems and costs.
- Solution will be very similar to the discrete case.
- The resulting **u** turns out to be a continuous state feedback rule.

Solution:

- The Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\mathbf{x}\mathbf{Q}\mathbf{x} + \frac{1}{2}\mathbf{u}^T\mathbf{R}\mathbf{u} + \lambda^T(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u})$$

- The end constraint on λ is

$$\lambda^T(t_f) = \frac{\partial \phi}{\partial \mathbf{x}(t_f)} \longrightarrow \lambda(t_f) = \mathbf{S}_f\mathbf{x}(t_f)$$

- λ's diff. eq. is

$$\dot{\lambda}^T(t) = -\frac{\partial \mathcal{L}}{\partial \mathbf{x}} - \lambda^T(t)\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \longrightarrow \dot{\lambda} = -\mathbf{Q}\mathbf{x} - \mathbf{A}^T\lambda$$

- The third Euler Lagrange condition gives

$$\frac{\partial \mathcal{H}}{\partial \mathbf{u}} = 0 \longrightarrow 0 = \mathbf{R}\mathbf{u} + \mathbf{B}^T\lambda$$

  Rearranging

$$\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^T\lambda$$

- This is a two point boundary value problem with **x** known at t_0 and λ known at t_f. The two linear differential equations are coupled by **u**.

- Plugging $\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^T\lambda$ into the state dynamics gives

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\lambda$$

- One can show (see Bryson & Ho, sec. 5.2) that there exists a time varying matrix **S**(t) that provides a linear relation between λ and **x**

$$\lambda(t) = \mathbf{S}(t)\mathbf{x}(t) \qquad \mathbf{S}(t_f) = \mathbf{S}_f$$

  Plugging this into the above state dynamics gives

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{S}(t)\mathbf{x}(t) \quad (*)$$

  i.e. $\mathbf{u} = \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{S}(t)\mathbf{x}(t)$ is a linear state feedback. Similarly to the discrete case, to know the control rule we simply need to find **S**(t).

- We plug $\lambda(t) = \mathbf{S}(t)\mathbf{x}(t)$ into $\dot{\lambda} = -\mathbf{Q}\mathbf{x} - \mathbf{A}^T\lambda$ and get

$$\dot{\mathbf{S}}\mathbf{x} + \mathbf{S}\dot{\mathbf{x}} = -\mathbf{Q}\mathbf{x} - \mathbf{A}\mathbf{S}\mathbf{x}$$

- Next we plug (*) into the above equation and after a bit of rearranging get

$$\left(\dot{\mathbf{S}} + \mathbf{S}\mathbf{A} + \mathbf{A}^T\mathbf{S} - \mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{S} + \mathbf{A}\right)\mathbf{x} = 0$$

- Since **x**(t)≠0 (otherwise there is no need for regulation) we can drop **x** and get...

… $$\dot{\mathbf{S}} = -\mathbf{S}\mathbf{A} - \mathbf{A}^T\mathbf{S} + \mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{S} - \mathbf{A}$$

- This is a quadratic differential equation in **S** with a boundary condition $\mathbf{S}(t_f) = \mathbf{S}_f$
- This equation is known as a matrix Riccati equation. It can be solved (e.g. by numeric integration from S_f backwards) to get $\mathbf{S}(t)$.
- Usually, this dynamic system is stable and reaches a steady state **S** as $t \to \infty$. The Matlab function `care` solves for S_∞ (if a real valued solution exists).
- S_∞ is good for regulating the system for a long duration (i.e. forever).
- Since this differential equation is quadratic, it may have more than one solution. The desired solution is PSD. Starting from **S**=0 (instead of $S=S_f$) and numerically integrating until convergence (i.e. till $\dot{\mathbf{S}} \to \mathbf{0}$) will give the PSD solution (see Bryson & Ho, sec. 5.4).
- We will see (in a future lecture) that the HJB equations show that $J=\mathbf{x}^T(t_0)\mathbf{S}(t_0)\mathbf{x}^T(t_0)$. (This is also true for the discrete case, were we used the notation **P** instead of **S**).

t_f given , $\mathbf{x}(0)$ & $\mathbf{x}(t_f)$ partly or fully given

- Same variation of J holds:

$$\delta J = \left[\left(\frac{\partial \phi}{\partial \mathbf{x}} - \lambda^T\right)\delta \mathbf{x}\right]_{t=t_f} + [\lambda^T \delta \mathbf{x}]_{t=t_0} + \int_{t_0}^{t_f} \left[\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}} + \dot{\lambda}^T\right)\delta \mathbf{x} + \frac{\partial \mathcal{H}}{\partial \mathbf{u}}\delta \mathbf{u}\right] dt$$

- Suppose $\mathbf{x}_k(t_f)$ is given, then $\delta\mathbf{x}_k(t_f) = 0$ and therefore we do not require

$$\frac{\partial \phi}{\partial \mathbf{x}_k(t_f)} - \lambda_k^T = 0$$

in order to zero out the variation

- Similarly, if $\mathbf{x}_k(t_0)$ is not given, then $\delta\mathbf{x}_k(t_0) \neq 0$ and to zero out δJ we require that

$$\lambda_k(t_0) = 0$$

- Note that if the system is not "controllable" the condition may be impossible to satisfy.

- The Euler Lagrange equations are now

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \\
\dot{\lambda}^T(t) &= -\frac{\partial \mathcal{H}}{\partial x} = -\frac{\partial \mathcal{L}}{\partial \mathbf{x}} - \lambda^T(t)\frac{\partial \mathbf{f}}{\partial \mathbf{x}}
\end{aligned}$$

$$\boxed{\lambda_k^T(t_f) = \frac{\partial \phi}{\partial x_k(t_f)} \quad or \quad x_k(t_f)\ given}$$

$$\frac{\partial \mathcal{H}}{\partial \mathbf{u}} = 0$$

$$\boxed{\forall k \quad \lambda_k(0) = 0 \quad or \quad x_k(0)\ given}$$

Minimum Jerk

- Find a path $\mathbf{x}(t)$, starting at rest (zero velocity & acceleration) from $\mathbf{x}_0$ at time t_0 and ending at rest at $\mathbf{x}_\mathrm{f}$ at time t_f so that the squared cumulative change in acceleration (jerk) along the path is minimal.
- Define the state to be $\mathbf{x} = [x, v, a]$ (position, velocity, acceleration) and the control signal to be $\mathbf{u}(t) = \dot{a}$
- The cost is:

$$J = \frac{1}{2} \int_{t_0}^{t_f} u^2(t) dt$$

  where $\mathcal{L} = \frac{1}{2} u^2$ and $\varphi = 0$
- We solve for the one dimensional (scalar position) case but solution holds for the multidimensional case as well.

- The dynamics equations are
$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)\\ f_1 &= \dot{x} = v\\ f_2 &= \dot{v} = a\\ f_3 &= \dot{a} = u\end{aligned}$$
- The Jacobian is therefore
$$\frac{\partial f}{\partial \mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$
- and (remembering $\mathcal{L} = \frac{1}{2}u^2$)
$$\frac{\partial L}{\partial \mathbf{x}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \frac{\partial f}{\partial \mathbf{u}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
- so, by $\dot{\lambda}^T(t) \quad = \quad -\frac{\partial \mathcal{H}}{\partial x} = -\frac{\partial \mathcal{L}}{\partial \mathbf{x}} - \lambda^T(t)\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ we get
$$[\dot{\lambda}_1, \dot{\lambda}_2, \dot{\lambda}_3]^T = \quad 0 - [\lambda_1, \lambda_2, \lambda_3] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[\dot\lambda_1, \dot\lambda_2, \dot\lambda_3]^T = [0, -\lambda_1, -\lambda_2]$$

- We can solve these differential equations:

$$\begin{aligned}\lambda_1 &= c_1\\ \lambda_2 &= -c_1 t + c_2\\ \lambda_3 &= \frac{1}{2}c_1 t^2 - c_2 t + c_3\end{aligned}$$

- The Hamiltonian (H = L+$\boldsymbol{\lambda}^{\mathrm{T}}\mathbf{f}$) is

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}u^2(t) + \lambda^T f\\ &= \frac{1}{2}u^2(t) + \lambda_1 v + \lambda_2 a + \lambda_3 u\end{aligned}$$

- so

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial \mathbf{u}} &= 0\\ 0 &= u + \lambda_3\\ u &= -\frac{1}{2}c_1 t^2 + c_2 t - c_3\end{aligned}$$

$$u = -\frac{1}{2}c_1 t^2 + c_2 t - c_3$$

- Plugging u into the dynamics equations:

$$\begin{aligned}
\dot{a} &= u \\
a &= -\frac{1}{6}c_1 t^3 + \frac{1}{2}c_2 t^2 - c_3 t + c_4 \\
\dot{v} &= a \\
v &= -\frac{1}{24}c_1 t^4 + \frac{1}{6}c_2 t^3 - \frac{1}{2}c_3 t^2 + c_4 t + c_5 \\
\dot{x} &= v \\
x &= -\frac{1}{120}c_1 t^5 + \frac{1}{24}c_2 t^4 - \frac{1}{6}c_3 t^3 + \frac{1}{2}c_4 t^2 + c_5 t + c_6
\end{aligned}$$

- Using the initial conditions $x(0) = x_0$ and $v(0) = a(0) = 0$ we see that $c_4 = c_5 = 0$ and $c_6 = x_0$ leaving us with

$$\begin{aligned}
a &= -\frac{1}{6}c_1 t^3 + \frac{1}{2}c_2 t^2 - c_3 t \\
v &= -\frac{1}{24}c_1 t^4 + \frac{1}{6}c_2 t^3 - \frac{1}{2}c_3 t^2 \\
x &= -\frac{1}{120}c_1 t^5 + \frac{1}{24}c_2 t^4 - \frac{1}{6}c_3 t^3 + x_0
\end{aligned}$$

- Using the final conditions, $x(t_f) = x_f$ and $v(t_f) = a(t_f) = 0$ we get 3 equations in 3 unknowns (with t_f as a parameter):

$$\begin{bmatrix} 0 \\ 0 \\ x_f - x_0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{6}t_f^3 & \frac{1}{2}t_f^2 & -t_f \\ -\frac{1}{24}t_f^4 & \frac{1}{6}t_f^3 & -\frac{1}{2}t_f^2 \\ -\frac{1}{120}t_f^5 & -\frac{1}{24}t_f^4 & -\frac{1}{6}t_f^3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

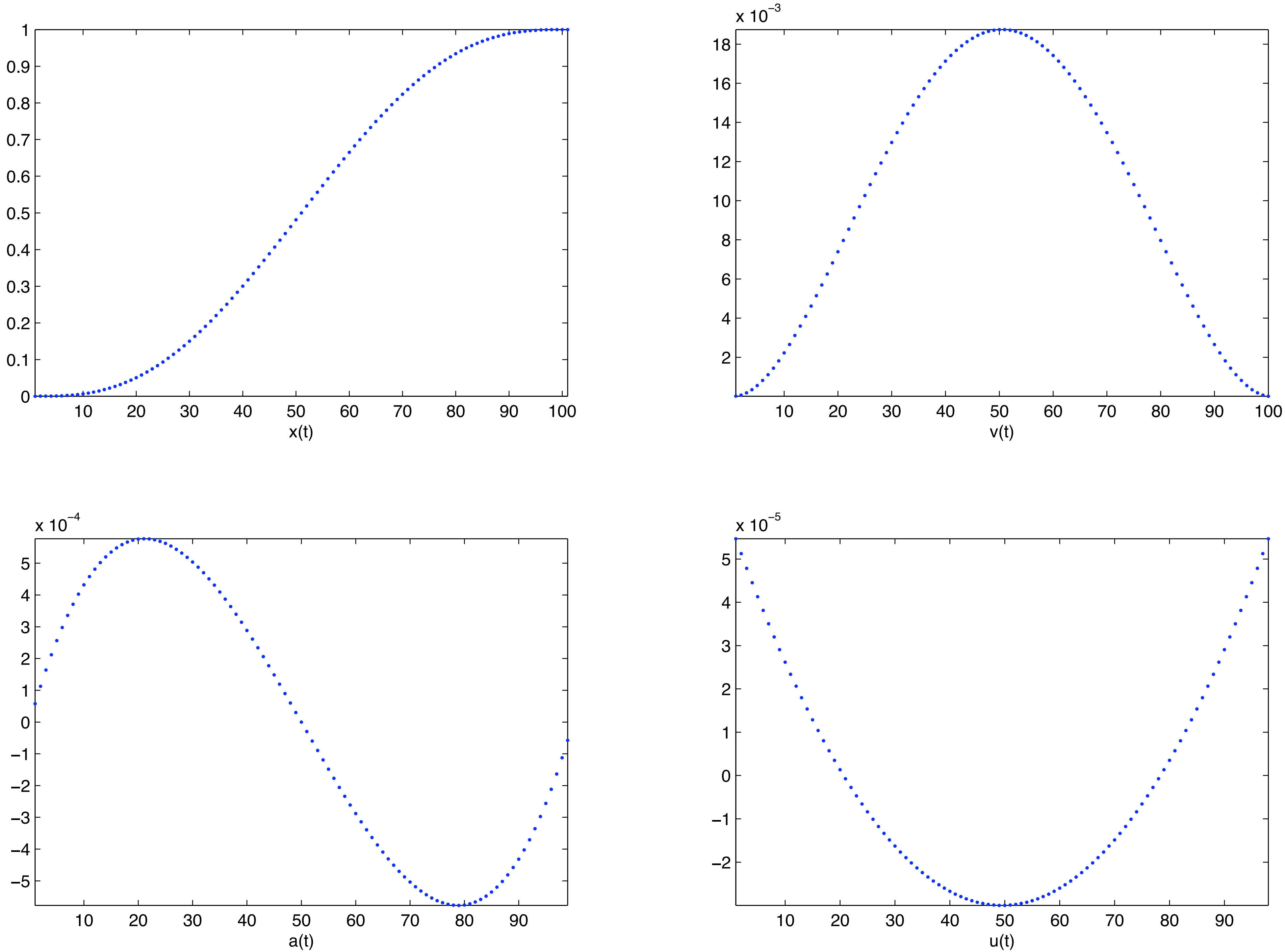
- The solutions of above equations yields:

$$c_3 = (x_f - x_0)\frac{60}{t_f^3}, \; c_2 = (x_f - x_0)\frac{360}{t_f^4}, \; c_1 = (x_f - x_0)\frac{720}{t_f^5}$$

- Resulting in

$$x(t) = x_0 + (x_f - x_0)(6\tau^5 - 15\tau^4 + 10\tau^3)$$

where $\tau = \frac{t}{t_f}$



x(t)
x 10^−3
v(t)
x 10^−4
a(t)
x 10^−5
u(t)

What If t_f Is Not Given?

- When t_f is not constrained it becomes a parameter of the variation problem and affects the cost of the solution. All the previous optimality conditions still apply and an extra condition is added:

$$\boxed{\left(\frac{\partial\phi}{\partial t} + \mathcal{H}\right)_{t=t_f} = \left(\frac{\partial\phi}{\partial t} + \mathcal{L} + \lambda^T \mathbf{f}\right)_{t=t_f} = 0}$$

Proof sketch (see Bryson & Ho, sec 2.7 for real proof):

- The Lagrange multipliers augmented cost is, as before

$$J_A = \phi\left(\mathbf{x}(t_f), t_f\right) + \int_{t_0}^{t_f} \mathcal{L}\left(\mathbf{x}(t), \mathbf{u}(t), t\right) dt + \int_{t_0}^{t_f} \lambda^T(t) \left[\mathbf{f}\left(\mathbf{x}(t), \mathbf{u}(t), t\right) - \dot{\mathbf{x}}(t)\right] dt$$

- The variation is now also in t_f. To simplify the proof we assume that $\mathbf{x}(t_f)$ is unchanged by the variation in t_f, that is $\mathbf{x}(t)=\mathbf{x}(t_f)$ for $t\in[t_f, t_f+\delta t]$.

- The resulting variation is

$$
\begin{aligned}
\delta J' &= \delta J + \left.\frac{\partial \phi}{\partial t}\right|_{t_f} \delta t_f + \mathcal{L}(t_f)\delta t_f + \lambda^T(t_f)\mathbf{f}(t_f)\delta t_f \\
&= \delta J + \underbrace{\left(\left.\frac{\partial \phi}{\partial t}\right|_{t_f} + \mathcal{L}(t_f) + \lambda^T(t_f)\mathbf{f}(t_f) \right)}_{require\ = 0} \delta t_f
\end{aligned}
$$

  where δJ is the variation given t_f

- Note that the same result is reached without the assumption on **x** not varying.

Minimum Time Problems

- This is an example of a common case where t_f is not given.
- problem setting: given dynamical system

$$\mathbf{u} \in \mathrm{R}^m \;, \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \in \mathrm{R}^n$$

  and some of the start/end state values $x_k(t_0)$ and $x_k(t_f)$ find the control that minimizes the following simple cost

$$J = t_f - t_0$$

- That is $\phi = 0 \quad , \quad \mathcal{L} = 1$

- The Euler Lagrange equations are

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \\
\dot{\lambda}^T &= -\lambda^T \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \\
\lambda_k(t_f) = 0 \quad &or \quad x_k(t_f)\ given, \quad k = \{1 \ldots n\} \\
\lambda^T \frac{\partial \mathbf{f}}{\partial \mathbf{u}} &= \mathbf{0} \quad m\ equations \\
\lambda_k(t_0) = 0 \quad &or \quad x_k(t_0)\ given, \quad k = \{1 \ldots n\} \\
\lambda^T(t_f)\mathbf{f}(t_f) &= -1
\end{aligned}
$$

- We have $2n$ boundary constraints for $2n$ differential equations, m optimality constraints for m control variables and one constraint for t_f.

Integral Constraint On Path

- Suppose we have an extra requirement from the optimal trajectory:

$$c = \int_{t_0}^{t_f} N(\mathbf{x}, \mathbf{u}, t)dt$$

- E.g. given fixed amount of fuel, reach the destination with an empty tank (while obeying some optimality criterions) . $N(\mathbf{x},\mathbf{u},t)$ would be the fuel consumption and c the given fuel amount.

- **Solution:** (by reduction to a regular problem with an extra state variable that has given boundary values)
  Add a new state variable, x_{n+1} with the following dynamics function f

$$\dot{x}_{n+1} = N(\mathbf{x}, \mathbf{u}, t)$$

- It is clear that

$$x_{n+1}(t') = \int_{t_0}^{t'} N(\mathbf{x}, \mathbf{u}, t)dt$$

- Now require that

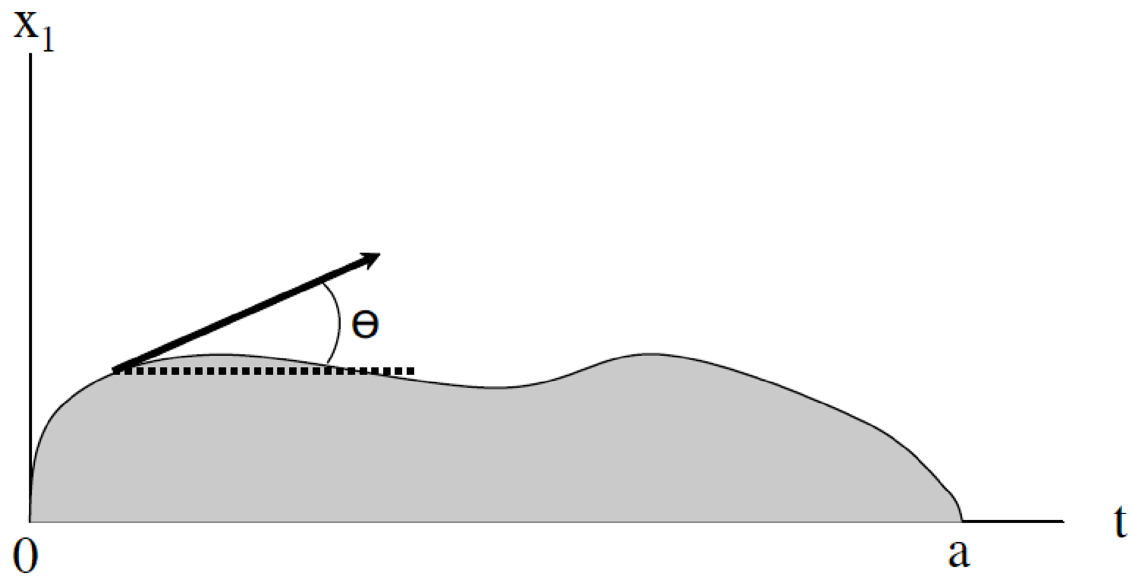
$$x_{n+1}(t_f) = c \quad , \quad x_{n+1}(t_0) = 0$$

- Obviously, the augmented system obeys the integral constraint

- (see Bryson & Ho Sec 3.1)

Integral Constraint Eg.

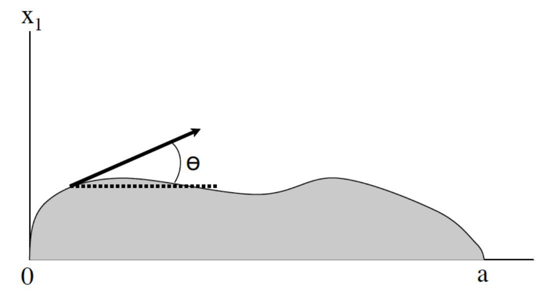
- **Maximum area under curve of given length:**



- We formulate it as a control problem: A particle moves with constant velocity (1) along the horizontal axis, starting at time 0 and ending at time *a*. The position along the x_1 axis changes as a function of the angle θ which is the control signal. The particle's path (in the x_1-t plane) must be p and the area under the particle is maximized.

- We assume that $-\pi/2<\theta<\pi/2$ (true if $p< \pi a$)

- The corresponding equations are



$$\mathcal{L} = -x_1 \quad , \quad \phi = 0$$

$$\begin{aligned} u &= \theta \\ f_1 &= \dot{x}_1 = \tan\theta \end{aligned}$$

$$x_1(0) = 0 \quad , \quad x_1(a) = 0$$

$$\begin{aligned} t_f &= a \\ p &= \int_0^a \frac{1}{\cos\theta} dt \end{aligned}$$

- The cost is

$$J = -\int_0^a x_1 dt$$

- Solution:
- Add another state variable, x_2, and replace the integral constraint with

$$f_2 = \dot{x}_2 = \frac{1}{\cos\theta} \quad , \quad x_2(0) = 0 \; , \; x_2(a) = p$$

- The Hamiltonian is

$$\mathcal{H} = \mathcal{L} + \lambda^T \mathbf{f} = -x_1 + \lambda_1 \tan\theta + \frac{\lambda_2}{\cos\theta}$$

we know that it is a constant because L and $\mathbf{f}$ are not explicit functions of time (see previous slide).

- The Lagrange multipliers obay

$$\begin{aligned} \dot{\lambda}_1 &= -\frac{\partial \mathcal{H}}{\partial x_1} = 1 \\ \dot{\lambda}_2 &= -\frac{\partial \mathcal{H}}{\partial x_2} = 0 \end{aligned} \longrightarrow \begin{aligned} \lambda_1 &= t + c_1 \\ \lambda_2 &= c_2 \end{aligned}$$

- The Lagrange equation for **u** gives

$$\frac{\partial \mathcal{H}}{\partial \mathbf{u}} = \frac{\partial \mathcal{H}}{\partial \theta} = \frac{\lambda_1}{\cos^2 \theta} + \lambda_2 \frac{\sin \theta}{\cos^2 \theta} = 0$$

so

$$\lambda_1 = -\lambda_2 \sin \theta = -c_2 \sin \theta$$

i.e

$$\boxed{\sin \theta(t) = \frac{-t - c_1}{c_2}}$$

- This means that under this control scheme the sine of θ (t) is linear.

- One can then use the given boundary conditions to find H,c_1,c_2, solve x_1 and show that the optimal path is an ark of a circle whose center is at $\left(\frac{a}{2}, -\frac{p \cos \alpha}{2\alpha}\right)$, its radius is $p/2$ and α obeys $\frac{\sin \alpha}{\alpha} = \frac{a}{p}$

Equality Constraints Over Functions of the State/Control

- Suppose we wish to find a trajectory that obeys an extra set of equality constraints, $\mathbf{c}(\mathbf{x},\mathbf{u},t)=\mathbf{0}$.
- This is no different than requiring that $\mathbf{f}(\mathbf{x}(t),\mathbf{u}(t),t) - \dot{\mathbf{x}}(t) = 0$.
- We therefore treat the new constraint in a similar manner. We rewrite the Hamiltonian to be:

$$\mathcal{H} = \lambda^T \mathbf{f} + \mathcal{L} + \nu^T \mathbf{c}$$

  where ν is an extra vector of Lagrange multipliers that gets the exact same treatment as λ.

- The rest of the Euler Lagrange constraint derivation is unchanged.
- (see Bryson & Ho, sec. 3.3)

Some Other Variation Variants

- *functions* of the state variables given at t_0 and/or given or unconstrained terminal time t_f. (Eg. manipulate a robotic hand from one curved wall to another along the shortest path).
- via point problems (trajectory is constrained to obey some rule at a certain time or position along its path).
- problems where the Lagrange equations do not produce a constraint on the control signal (although it is obviously constrained). Eg. bang-bang control.
- ...