Digital Signal Processing formula sheet

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1 introductory stuff

1.1 even & odd signals

$x_e(n) = \frac{1}{2}\big(x(n) + x^*(-n)\big) \qquad x_o(n) = \frac{1}{2}\big(x(n) - x^*(-n)\big)$

1.2 sum of sequence

$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \qquad \sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha} \quad (\forall \alpha \neq 1) \qquad \sum_{n=M}^{N-1} \alpha^n = \frac{\alpha^M - \alpha^N}{1-\alpha}$

1.3 unit sample synthesis

$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$

1.4 representation by impulse response

for an LTI system T:

$$y(n) = \mathrm{T}\big[x(n)\big] = \mathrm{T}\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right] = \sum_{k=-\infty}^{\infty} x(k) \underbrace{\mathrm{T}\big[\delta(n-k)\big]}_{impulse\ response} = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = x(n) \star h(n)$$

1.4.1 stability

definition: $|x(n)| < \infty \Longrightarrow |y(n)| < \infty$
condition: BIBO $\iff \sum_{n=-\infty}^{\infty} |h(n)| < \infty$ for proof of $\Longrightarrow$ use $X(k) = \frac{h^*(-k)}{|h(-k)|}$ (remember $y(0) < \infty$)

1.4.2 casual system

casual $\iff h(n) = 0 \quad \forall n < 0$

1.5 representation by difference equation:

$\sum_{k=0}^{N} a_k y(n-k) = \sum_{m=0}^{M} b_m x(n-m)$ or: $y(n) = \sum_{m=0}^{M} b_m x(n-m) - \sum_{k=1}^{N} a_k y(n-k)$

1.5.1 solution:

1. $y(n) = y_H(n) + y_P(n)$
   $y_H(n) = \sum_{k=1}^{N} c_k \lambda_k^n$ $\qquad \lambda_i$ are roots of $\sum_{k=0}^{N} a_k \lambda^{N-k}$ (for each identical λ_i multiply by n).
2. $y(n) = y_{ZI} + y_{ZS}$
   initial conditions are:
   zero state: $y(n) = 0 \quad -N \leq n \leq -1$
   zero input: $x(n) = 0 \quad -M \leq n \leq -1$

1.6 FIR–finite duration impulse response

$y(n) = \sum_0^M b_m x(n-m) \quad h(n) = b_n$

1.7 IIR–infinite duration impulse response

example: $\sum_0^M a_k y(n-k) = x(n)$

2 Fourier Series

$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi}{T}kt}$
$c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-j\frac{2\pi}{T}kt}$

3 CTFT - continuous time fourier transform

$X(\omega) = \mathcal{F}\left[x(t)\right] \triangleq \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$
example: $\mathcal{F}\left[\text{sinc}(\frac{\pi}{T}t)\right] = T\dot{\text{r}}\text{ect}[-\frac{\pi}{T}, \frac{\pi}{T}]$ $\quad\mathcal{F}\left[e^{-j\omega_0 n}\right] = 2\pi\delta(\omega + \omega_0)$

4 DTFT

$X(\omega) \triangleq \mathcal{F}\left[x(n)\right] = \sum_{-\infty}^{\infty} x(n)e^{-j\omega n}$ (only if $\sum_{-\infty}^{\infty}|x(n)| < \infty$)

4.1 properties of DTFT

1. **periodicity** $X(\omega) = X(\omega + 2\pi) \Longrightarrow$ We need to know $X(\omega)$ only for $\omega \in [0, 2\pi]$
2. **Implication** $X(\omega) = X^*(-\omega) \Longrightarrow$ need only $\omega \in [0, \pi]$
3. **Linearity**
4. **time shift** $\mathcal{F}\left[x(n-k)\right] = X(\omega)e^{-j\omega k}$
5. **freq shift** $\mathcal{F}\left[x(n)e^{j\omega_o n}\right] = X(\omega - \omega_0)$
6. **conjugation** $\mathcal{F}\left[x^*(n)\right] = X^*(e^{-j\omega})$
7. **Folding** $\mathcal{F}\left[x(-n)\right] = X(-\omega)$
8. **Symetries in real sequences** $x(n) = x_e(n) + x_o(n)$
   $Re\left(X(\omega)\right) = \mathcal{F}\left[x_e(n)\right]$ $\quad Im\left(X(\omega)\right) = \mathcal{F}\left[x_o(n)\right]$
9. **convolution** $\mathcal{F}\left[x_1(n) * x_2(n)\right] = X_1(\omega)X_2(\omega)$
10. **Modulation (dual convolution)** $\mathcal{F}\left[x_z(n)x_2(n)\right] = \mathcal{F}\left[x_1(n)\right] * \mathcal{F}\left[x_2(n)\right]$
11. **Energy (Parseval's theorem)** $energy \triangleq \sum_{n=-\infty}^{\infty}|x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|X(\omega)|^2 d\omega$

5 IDTFT

$x(n) \triangleq \mathcal{F}^{-1}\left[X(\omega)\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$

6 DFT

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n)e^{-j\omega_k n} & \omega_k &\triangleq \frac{2\pi k}{N} \\ &= \sum_{n=0}^{N-1} x(n)W_N^{-kn} & W_N &\triangleq e^{\frac{j2\pi}{N}} \\ &= Ax(n) & A_{N(n,k)} &\triangleq (W_N)^{-nk} \end{aligned}$$

IDFT: $x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j\omega_k n}$

example: for $N = 4$, $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$

6.1 Example for DFT evaluation

$x(n) = \begin{cases} 1 & mN \le n \le mn + L - 1 \\ 0 & mN + L \le n \le (m+1)N - 1 \end{cases}$ (periodic square wave with $\frac{L}{N}$ duty cycle)

$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}nk} = \sum_{n=0}^{L-1}\left(e^{-j\frac{2\pi}{N}}\right)^n = \begin{cases} L & k = 0, \pm N, \pm 2N, \cdots \\ \frac{1-e^{-j2\pi Lk/N}}{1-e^{-j2\pi k/N}} & \text{otherwise} \end{cases}$

in the 'otherwise' option: $= \frac{e^{-j\pi Lk/N}}{e^{-j\pi k/N}} \cdot \frac{e^{j\pi Lk/N} - e^{-j\pi Lk/N}}{e^{j\pi k/N} - e^{-j\pi k/N}} = e^{\frac{-k\pi(L-1)k}{N}} \cdot \frac{\sin(\frac{\pi kL}{N})}{\sin(\frac{\pi k}{N})}$

6.2 relation between DFT and DTFT

define $\omega_0 \triangleq \frac{2\pi}{N} \Longrightarrow X(k) = X(\omega)\mid_{\omega = k\omega_0} \Longrightarrow$ calculation of DFT by sampling DTFT at $k\omega_0 \quad k \in [1, N]$

6.3 theorem: Frequency sampling

if $x(n)$ is time limited to $[0, N-1]$, then N samples of $X(z)$ on the unit circle determines $x(n) \Longrightarrow$ determines $X(z) \quad \forall z$.

6.4 Reconstruction from DFT

from DFT to $\mathcal{Z}$: $X(z) = \frac{1-z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1-e^{j\frac{2\pi}{N}k} z^{-1}}$

from DFT to DTFT: $X(\omega) = \sum_{k=0}^{N-1} X(k) \Phi\left(\omega - \frac{2\pi k}{N}\right) \quad \Phi(\omega) \triangleq \frac{\sin\left(\frac{\omega N}{2}\right)}{N \sin\left(\frac{\omega}{2}\right)} \cdot e^{-j\omega\left(\frac{N-1}{2}\right)}$

6.5 circular convolution

$x_1(n) \circledast x_2(n) \triangleq \sum_{m=0}^{N-1} x_1(m) x_2\left((n-m) mod\, N\right)$ $\qquad$ N is the dimension of the bigger signal.

7 $\mathcal{Z}$ transform

7.1 definitions

$X(Z) \triangleq \mathcal{Z}\left[x(n)\right] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$x(n) \triangleq \mathcal{Z}^{-1}\left[X(z)\right] = \frac{1}{2\pi j} \oint_c X(z) z^{n-1} dz$

7.2 properties of $\mathcal{Z}$ transform

1. **Linearity**
2. **Sample shifting (time shifting):**
   (a) $\mathcal{Z}\left[x(n-n_0)\right] = z^{-n_0} X(z)$ $\qquad$ ROC unchanged
   (b) $\mathcal{Z}^+[x(n-k)] = \sum_{m=-k}^{-1} x(m) z^{-(m+k)} + z^{-k} X^+(z)$
3. **Frequency shifting:** $\mathcal{Z}\left[a^n x(n)\right] = X(\frac{z}{a})$ $\qquad$ $ROC = ROC_x$ scaled by $|a|$
4. **Folding:** $\mathcal{Z}\left[x(-n)\right] = X(\frac{1}{z})$ $\qquad$ $ROC =$ inverted ROC_x
5. **Complex conjugation:** $\mathcal{Z}\left[x^*(n)\right] = X^*(z^*)$ $\qquad$ ROC unchanged
6. **Differentiation in the z domain:** $\mathcal{Z}\left[nx(n)\right] = -z \frac{d}{dz} X(z)$ $\qquad$ ROC unchanged
   or: $(-1)^k z^k \frac{d^k X(z)}{dz^k} = \mathcal{Z}\left[\frac{(n+k-1)!}{(n-1)!} x(n)\right]$
7. **Multiplication:** $\mathcal{Z}\left[x_1(n) x_2(n)\right] = \frac{1}{2\pi j} \oint_c X_1(\nu) X_2(\frac{z}{\nu}) \nu^{-1} d\nu$ $\qquad$ $ROC = ROC_{x1} \cap invertedROC_{x2}$
8. **convolution:** $\mathcal{Z}\left[x_1(n) * x_2(n)\right] = X_1(z) X_2(z)$ $\qquad$ $ROC = ROC_{x1} \cap ROC_{x2}$

7.3 example of properties usage

$x(n) = (n-2)(0.5)^{n-2} \cos[\frac{\pi}{3}(n-2)] u(n-2)$

$\mathcal{Z}\left[x(n)\right] \overset{2}{=} z^{-2} \mathcal{Z}\left[n(0.5)^n \cos\frac{\pi n}{3} u(n)\right] \overset{6}{=} z^{-2}\left(-z \frac{d}{dz} \mathcal{Z}\left[(0.5)^n \cos(\frac{\pi n}{3}) u(n)\right]\right) = \cdots$

7.4 $\mathcal{Z}$ transforms

	$x(n)$	$\mathcal{Z}\left[x(n)\right]$	ROC
1.	$\delta(n)$	1	$\forall z$
2.	$u(n)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3.	$-u(-n-1)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4.	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
5.	$-b^n u(-n-1)$	$\frac{1}{1-bz^{-1}}$	$\lvert z\rvert < \lvert b\rvert$
6.	$[a^n \sin(\omega_0 n)] u(n)$	$\frac{(a \sin(\omega_0)) z^{-1}}{1-(2a\cos(\omega_0)) z^{-1} + a^2 z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
7.	$[a^n \cos(\omega_0 n)] u(n)$	$\frac{1-(a\cos(\omega_0)) z^{-1}}{1-(2a\cos(\omega_0)) z^{-1} + a^2 z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
8.	$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
9.	$-nb^n u(-n-1)$	$\frac{bz^{-1}}{(1-bz^{-1})^2}$	$\lvert z\rvert < \lvert b\rvert$

7.5 comments about ROC

ROC $= (R_-, R_+)$ $\qquad$; casual $\Longrightarrow R_+ = \infty$ $\qquad$; non-casual $\Longrightarrow R_- = 0$

7.6 Usages of $\mathcal{Z}$ transform

7.6.1 System representation in $\mathcal{Z}$ domain

by $\mathcal{Z}$ transforming a given difference equation: $y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{m=0}^{M} b_m x(n-m)$

$$H(z) \triangleq \frac{Y(z)}{X(z)} = \frac{\sum_{l=0}^{M} b_l z^{-l}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = b_0 z^{N-M} \frac{\prod_{l=1}^{M}(z - z_l)}{\prod_{k=1}^{N}(z - p_k)}$$

7.6.2 frequency response

if unit circle lies within the ROC, then: $frequency\ response = H(\omega) = b_0 e^{j\omega(N-M)} \frac{\prod_{l=1}^{M}(e^{j\omega} - e_l^z)}{\prod_{k=1}^{N}(e^{j\omega} - p_k)}$

magnitude: $|H(\omega)| = |b_0| \frac{|e^{j\omega} - z_1| \cdots |e^{j\omega} - z_M|}{|e^{j\omega} - p_1| \cdots |e^{j\omega} - p_N|}$

phase: $\angle H(\omega) = \underbrace{[0\ or\ \pi]}_{\text{constant}} + \underbrace{[N-M]\omega}_{\text{linear}} + \underbrace{\sum_{1}^{M} \angle(e^{j\omega} - z_k) - \sum_{1}^{M} \angle(e^{j\omega} - p_k)}_{\text{nonlinear}}$

7.6.3 example: solution of difference equations

$y(n) - \frac{3}{2}y(n-1) + \frac{1}{2}y(n-2) = x(n) \qquad y(-1) = 4 \quad y(-2) = 10 \quad x(n) = (\frac{1}{4})^n u(n)$

$Y^+(z) - \frac{3}{2}\left[y(-1) + z^{-1}Y^+(z)\right] + \frac{1}{2}\left[y(-2) + z^{-1}y(-1) + z^{-2}Y^+(z)\right] = \frac{1}{1-\frac{1}{4}z^{-1}}$

$\Longrightarrow Y^+(z) = \underbrace{\frac{\frac{1}{1-\frac{1}{4}z^{-1}}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}}_{\text{zero state}} + \underbrace{\frac{1 - 2z^{-1}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}}_{\text{zero input}} = \frac{2 - \frac{9}{4}z^{-1} + \frac{1}{2}z^{-2}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})(1-\frac{1}{4}z^{-1})} = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{\frac{2}{3}}{1-z^{-1}} + \frac{\frac{1}{3}}{1-\frac{1}{4}z^{-1}}$

$\Longrightarrow y(n) = \left[\left(\frac{1}{2}\right)^n + \frac{2}{3} + \frac{1}{3}\left(\frac{1}{4}\right)^n\right] u(n)$

7.6.4 stability

theorem: LTI stable $\Longleftrightarrow$ unit circle $\subset$ ROC   casual LTI stable $\Longleftrightarrow$ all poles inside unit circle.

7.7 summary

from a representation to other	how to do it
$h(n) \longleftrightarrow H(z)$	$\mathcal{Z}$ transform.
$H(z) \longrightarrow$ transfer function $(H(\omega))$	$z = e^{i\omega}$
Difference equations $\longrightarrow H(z)$	$\mathcal{Z}$ transform, then solve for $\frac{Y}{X}$
$H(z) \longrightarrow$ difference equations	Like in the example coming up.
Difference equation $\longrightarrow H(\omega)$	DTFT, then solve for $\frac{Y}{X}$
$h(n) \longleftrightarrow H(\omega)$	DTFT (if exists).

7.8 example: $H(z) \longrightarrow$ difference equation

$H(z) = \frac{z+1}{z^2 - 0.9z + 0.81} \quad \Longrightarrow \frac{Y(z)}{X(z)} = \frac{z+1}{z^2-0.9z+0.81}\left(\frac{z^{-2}}{z^{-2}}\right) = \frac{z^{-1}+z^{-2}}{1-0.9z^{-1}+0.81z^{-2}}$

$\Longrightarrow Y(z) - 0.9z^{-1}Y(z) + 0.81z^{-2}Y(z) = z^{-1}X(z) + z^{-2}X(z)$

$\Longrightarrow y(n) - 0.9y(n-1) + 0.81y(n-2) = x(n-1) + x(n-2)$

8 Filter Design

8.1 FIR

$y(n) = b_0 x(n) + b_1 x(n-1) + \cdots + b_{M-1}x(n-M+1) \qquad$ length $= M \qquad$ order $= M-1$

$H(z) = b_0 + b_1 z^{-1} + \cdots + b_{M-1}z^{-(M-1)}$

$h(n) = \begin{cases} b_n & 0 \le n \le M-1 \\ 0 & \text{else} \end{cases}$

8.2 Linear phase form

$\angle H(\omega) = \beta - \alpha\omega \quad -\pi < \omega \le \pi \quad \beta \in \{0, \pm\frac{\pi}{2}\} \quad \alpha$ constant

symetric impulse response: $h(n) = h(M-1-n) \quad \beta = 0 \quad 0 \le n \le M-1$

anti-symetric impulse response: $h(n) = -h(M-1-n) \quad \beta = \pm\frac{\pi}{2} \quad 0 \quad 0 \le n \le M-1$

8.2.1 Type 1: Symmetrical impulse response, M odd

$H(\omega) = \left[\sum_{n=0}^{\frac{M-1}{2}} a(n)\cos\omega n\right] e^{-j\omega\frac{M-1}{2}}$
the middle sample $a(0) = h(\frac{M-1}{2})$ $\quad a(n) = 2h(\frac{M-1}{2} - n), \quad 1 \le n \le \frac{M-3}{2}$

8.2.2 Type 2: Symmetrical impulse response, M even

$H(e^{j\omega}) = \left[\sum_{n=1}^{M/2} b(n)\cos\{\omega(n-\frac{1}{2})\}\right] e^{-j\omega\frac{(M-1)}{2}}$
$b(n) = 2h(\frac{M}{2} - n)$, n = 1,2,..., $\frac{M}{2}$ $\quad$ note: at $\omega = \pi$ we get $H_r(\pi) = 0$

8.2.3 Type 3: Antisymmetric impulse response, M odd

$H(e^{j\omega}) = \left[\sum_{n=1}^{\frac{M-1}{2}} c(n)\sin\omega n\right] e^{j[\frac{\pi}{2}-(\frac{M-1}{2})\omega]}$
$c(n) = 2h(\frac{M-1}{2} - n) \quad n = 1, \ldots, \frac{M-1}{2}$

8.2.4 Type 4: Antisymmetric impulse response, M even

$H(e^{j\omega}) = \left[\sum_{n=1}^{M/2} d(n)\sin\{\omega(n-\frac{1}{2})\}\right] e^{j[\frac{\pi}{2}-\omega\frac{M-1}{2}]}$
$d(n) = 2h(\frac{M}{2} - n)$, n = 1,2,..., $\frac{M}{2}$ note: At $\omega = 0$, $H_r(0) = 0$

8.3 IIR

BiLinear: $\omega = 2\arctan(\frac{\Omega T}{2})$ $\quad s = \frac{2}{T}\frac{z-1}{z+1}$ $\quad z = \frac{1+sT/2}{1-sT/2}$
$r = 1 - \frac{BW}{FS}\cdot\pi$
$\Omega = \frac{2}{T}\tan(\frac{\omega}{2})$
$n = \frac{\log(10^{-Gs/10}-1)/(10^{-Gp/10}-1)}{2\log(\Omega s/\Omega p)}$
$\Omega_c = \frac{\omega p}{[10^{-Gp/10}-1]^{1/2n}}$ $\quad$ you can replace p with s
$H_a(s) = \frac{\Omega_c^N}{\prod(s-Pk)}$
$P_k = \Omega c e^{j\frac{\Pi}{2N}(2k+N+1)} k = 0 \ldots N-1$
A/D: $H_a(s) \longrightarrow H_a(z) = \frac{c}{s-p} = \frac{c}{1-e^{pT}z^{-1}}$ $\quad$ ($2\pi/T$ can be added to each s)
$|H_a(\Omega)^2| = \frac{1}{1+(\frac{\Omega}{\Omega_c})^{2N}}$

8.4 Frequency sampling form

$H(z) = \mathcal{Z}[h(n)] = \mathcal{Z}[\text{IDFT}\{H(k)\}] = \left(\frac{1-z^{-M}}{M}\right)\sum_{k=0}^{M-1}\frac{H(k)}{1-W_M^k z^{-1}}$ $\quad W_M = e^{\frac{2\pi j}{M}}$

9 Windows (not Bill Gates' Windows!)

9.1 introduction

windowing gives us a sinc around every ω in the original signal. That is, if the original signal had angular frequencies of ω_1 and ω_2, then at each of these frequencies we will have a sinc in the Fourier graph.

9.2 The problem:

2 close frequencies may have an overlapping sinc.

9.3 The condition for separation of frequencies:

The main lobe of the sinc cuts the ω axis at $\pm\frac{2\pi}{M}$ (and then every $\pm\frac{2\pi}{M}$ again). Its width is $\frac{4\pi}{M}$. We want each of the two ω to have that 0 before the other one begins its main lobe. So: $\Delta\omega \triangleq |\omega_1 - \omega_2| > \frac{4\pi}{M}$

9.4 Hanning window:

It has its main lobe width $\frac{8\pi}{M}$, but the ratio between the main lobe and second lobe is much better (so a main lobe of other frequency will not be mistaken to a second lobe of the former frequency). The condition for separation will thus become: $|\omega_1 - \omega_2| > \frac{8\pi}{M}$

9.5 Windows properties

type	$\|\omega_1 - \omega_2\|$	lobe size	comments
square	4π	$-13dB$	
hamming	8π	$-25dB$	equiripple
hanning	8π	$-31dB$	
b/a	12π	$-51dB$	

10 The sampling theory - an intuitive approach

10.1 definitions

F_0 = maximum frequency in the sampled signal (sometimes called **band width**
Ω is the radian frequency in the **analog** signal.
ω is the radian frequency in the **digital** signal.
$\mathcal{F}(\omega)$ = Fourier transform of sampled signal (digital signal).
$\mathcal{F}(\Omega)$ = Fourier transform of analog signal.
F_s = frequency of sampling. T = period of sampling = $\frac{1}{F_s}$
important formula: $\omega = \Omega \cdot T$

10.2 observations of the Fourier transform of sampled signal:

When comparing $\mathcal{F}(\omega)$ to $\mathcal{F}(\Omega)$ on the same axis (the frequency axis is of Ω then:

1. The amplitude of $\mathcal{F}(\omega)$ is $\frac{1}{T} \cdot \mathcal{F}(\Omega)$.
2. The frequency of $\mathcal{F}(\Omega)$ is scaled by T in the $\mathcal{F}(\omega)$ image (i.e. usually this results in a narrowing of the $\mathcal{F}(\Omega)$ image, since usually $T < 1$).
3. There are replicas every $\frac{2\pi}{T}$ in the $\mathcal{F}(\omega)$ graph, and the central one is around the origin (and therefore the 2nd replica starts at π/T).

10.3 conclusions

If we narrow our $\mathcal{F}(\omega)$ enough so that it will 'finish' before we get to π/T, then we will not lose information due to the replicas aliasing. So we want the maximal Ω to be less than π/T
$\Longrightarrow \Omega_{\max} = 2\pi F_0 < \pi/T = \pi F_s \Longrightarrow F_s > 2 \cdot F_0$ (the Nyquist frequency).

11 The sampling theory - a more formal approach

11.1 Poisson theorem

$$c(t) = \Sigma_{n=-\infty}^{n=\infty}\delta(t - nT) \Longleftrightarrow C(\omega) = \Sigma_{n=-\infty}^{n=\infty}e^{-j\omega nT} = \frac{2\pi}{T}\Sigma_{k=-\infty}^{k=\infty}\delta(\omega - \frac{2\pi k}{T})$$

11.2 the actual sampling:

The sampled signal(still in continuous time): $\widetilde{x_d}(t) = \Sigma_{n=-\infty}^{n=\infty}x(nT)\delta(t - nT)$
CTFT (using poisson theorem): $\widetilde{X_d}(\Omega) = \frac{1}{T}\Sigma_{n=-\infty}^{n=\infty}X(\Omega - \frac{2\pi k}{T})$
The discrete sampled signal: $x_d(n) = x(nT)$
DTFT: $X_d(\omega) = \frac{1}{T}\Sigma_{n=-\infty}^{n=\infty}X(\frac{\omega}{T} - \frac{2\pi k}{T})$

11.3 The reconstruction theorem:

$supp(X(\omega)) \in [\frac{-\pi}{T}, \frac{\pi}{T}]$
$x(t) = \Sigma_{n=-\infty}^{n=\infty}x(nT)h_T(t - nT)$ **where:** $h_T = \frac{T\sin\frac{\pi T}{T}}{\pi T} = sinc(\frac{\pi T}{T})$

12 Units

T	sec/sam
ω	rad/sam
Ω	rad/sec
f	Hz = cycle/sec
F_s	sample/sec

$$2\pi f = \Omega = \frac{\omega}{T} = \omega F_s$$