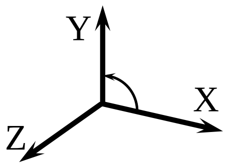
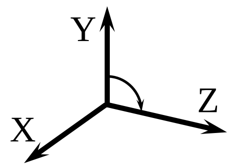
Transformations in 3D

3D Coordinate Systems

- Right-handed coordinate system:



- Left-handed coordinate system:



3D Transformations

- A point is represented by a 3D column vector: $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$
- Homogeneous coordinates: $\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$
- Transformations are 4 by 4 matrices:

$$\begin{bmatrix} a & b & c & t_x \\ d & e & f & t_y \\ g & h & i & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3D Transformations

- Translation: $\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ z+c \\ 1 \end{bmatrix}$

- Scaling: $\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} ax \\ by \\ cz \\ 1 \end{bmatrix}$

3D Shearing

- Shearing:

$$\begin{bmatrix} 1 & a & b & 0 \\ c & 1 & d & 0 \\ e & f & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + ay + bz \\ cx + y + dz \\ ex + fy + z \\ 1 \end{bmatrix}$$

- The change in each coordinate is a linear combination of all three
- Transforms a cube into a general parallelepiped

3D Rotation

- Rotation about the *x*-axis:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y\cos\theta - z\sin\theta \\ y\sin\theta + z\cos\theta \\ 1 \end{bmatrix}$$

- Rotations about each of the other two axes are defined similarly.
- Rotations are orthogonal matrices, preserving distances and angles.

Rotation About an Arbitrary Axis

- The idea: make the arbitrary axis coincident with one of the coordinate axes, rotate, and then transform back:
  - Translate rotation axis to pass through the origin;
  - Rotate about the X axis into the XZ plane;
  - Rotate about the Y axis into the YZ plane - rotation axis is now aligned with the Z axis;
  - Rotate about the Z axis by the desired angle;
  - Apply inverse rotations about the Y and X axes;
  - Apply inverse translation.

3D Reflection

- Through the *xy* plane: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- Reflections through the *xz* and the *yz* planes are defined similarly
- How can we reflect through some arbitrary plane?

Transforming Planes

- One way to transform a plane is by transforming any three non-collinear points on the plane.
- Another way is to transform the plane equation directly: $Ax + By + Cz + D = \begin{bmatrix} A & B & C & D \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0$
- Given a transformation T that transforms [*x*,*y*,*z*] to [*x′*,*y′*,*z′*] find *A′*, *B′*, *C′*, and *D′*, such that:

$$\begin{bmatrix} A' & B' & C' & D' \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = 0$$

(continued)

- Note that $\begin{bmatrix} A & B & C & D \end{bmatrix} T^{-1} T \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0$
- Thus, the transformation that we should apply to the plane equation is:

$$\left(T^{-1}\right)^T \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} A' \\ B' \\ C' \\ D' \end{bmatrix}$$