Basic Geometry Review

Basic Geometric Entities

- Scalars - real numbers
  - sizes/lengths, angles
- Vectors - typically 2D, 3D, 4D
  - directions
- Points - typically 2D, 3D, 4D
  - locations

Spaces

- Scalar field: formed by scalars and the operations between them (+,*).
- Vector space: formed by vectors, scalars, and the operations between them.
- Note: a purely abstract vector space has no notion of: distance, size, angle, or point!

Affine spaces

- An affine space adds the notion of point to the vector space: A = (P,V), where V is a vector space and P is a set of points:
  - for every point p and vector v, p+v $\in$ P
  - for every two points p,q: p-q $\in$ V.
- By choosing a basis and designating an origin point, we define an affine frame:

$$\mathbf{p} = \mathbf{a} + \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$$

Euclidean / Cartesian spaces

- A Euclidean space is an affine space with a distance metric based on inner product:

$$|\mathbf{a}-\mathbf{b}| = \sqrt{\langle \mathbf{a}-\mathbf{b}, \mathbf{a}-\mathbf{b} \rangle} = \sqrt{(\mathbf{a}_1-\mathbf{b}_1)^2 + \cdots + (\mathbf{a}_n-\mathbf{b}_n)^2}$$

- Cartesian space: Euclidean space with a standard orthonormal frame:
  - Basis vectors have length 1
  - Basis vectors are pairwise perpendicular (orthogonal)

Operations on Vectors

- Addition

$$\mathbf{u}+\mathbf{v} = [u_1+v_1, \ldots, u_n+v_n]^{\mathrm{T}}$$

- Multiplication by a scalar

$$s\mathbf{u} = [su_1, \ldots, su_n]^{\mathrm{T}}$$

- Dot product (scalar product)

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}^{\mathrm{T}}\mathbf{v} = u_1v_1 + \cdots + u_nv_n$$

Operations on Vectors

- Cross product (vector product)

$$\mathbf{u}\times\mathbf{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

- What is the geometric meaning of these operations?

Operations on Vectors

- Vector size/length/norm:

$$|\mathbf{v}| = \sqrt{\mathbf{v}_1^2 + \cdots + \mathbf{v}_n^2}$$

- Unit vector = pure direction $|\mathbf{v}| = 1$
- General vector = size * direction
- Vector normalization: $\mathbf{v} \leftarrow \frac{\mathbf{v}}{|\mathbf{v}|}$

Operations on Points

- Can't add points…
  - but can add a vector to a point $\mathbf{a} + \mathbf{v} = \mathbf{b}$
  - also can subtract points $\mathbf{v} = \mathbf{b} - \mathbf{a}$

- Can't multiply a point by a scalar…
  - but can take the affine combination of two points: $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$ where $s + t = 1$
  - can also take the affine combination of n > 2 points.

Lines and Planes

- A line in 2D can be defined by:
  - all affine combinations of two points
  - implicit equation: $f(x,y) = ax+by+c = 0$
  - parametric equation: $L(t) = O + tD$

- A plane can be defined by:
  - all affine combinations of three points
  - implicit equation: $f(x,y,z) = ax+by+cz+d = 0$
  - parametric equation: $P(s,t) = O + sD_1 + tD_2$