The Complexity of the Homotopy Method, Equilibrium Selection, and Lemke-Howson Solutions

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new result (with Papadimitriou and Savani): it is \textbf{PSPACE}-complete to find any of the equilibria that are computed by the Lemke-Howson algorithm (and related homotopy methods).
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- homotopy methods, what they are
- some proof ideas (with nice diagrams)
- related open questions
Suppose you want to solve the above game $G$
The homotopy method

Follow path of Nash equilibria along continuum of games: 
\[ \{ G_t : t \in [0, 1] \} \].

\[ G_1 = G; \ G_0 \text{ has “obvious” solution; } G_t = (1 - t)G_0 + tG. \]
Browder’s fixed point theorem

Given a continuum of continuous functions $f_t : D \rightarrow D$ on a compact domain $D$, $t \in [0, 1]$, there is a path of fixpoints in $D \times [0, 1]$ that connects a fixpoint of $f_0$ with a fixpoint of $f_1$.

Thanks to

(and the paper: On continuity of fixed points under deformations of continuous mappings, Summa Brasiliensis Math 4, 183-191 (1960))

F.E. Browder
The homotopy method: why it works

Browder’s fixed point theorem

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So, if in fact $f_0$ has a single unique fixpoint, starting there you end at a fixpoint of $f_1$. 

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Applied to games, the continuous functions are the “incentive direction” functions.

The homotopy approach is a “natural” idea. $G_0$ can be considered as a “prior belief” that helps select particular equilibria.

But notice that the path is not monotonic in $t$.

**LINEAR TRACING**

compute the NE resulting from game $G_0$ where each player gets a reward of 1 for his first strategy, 0 for the others.

Other homotopies are possible, including...
The Lemke-Howson algorithm (1964)

- $G_0$ is derived from $G$ be awarding a large bonus $B$ to one of the players (say, the row player) for choosing some specific pure strategy.
- For sufficiently large $B$, $G_0$ has an obvious solution.
- $G_t$ is similar, using a bonus of $(1 - t)B$, so $G_t$ interpolates continuously between $G_0$ and $G = G_1$.

The Browder path is piecewise linear; linear pieces connected by the pivot operations of the algorithm. Strategy that gets bonus corresponds to initially-dropped label.
Given a graph $G$ of indegree/outdegree at most 1, and a vertex of degree 1, find another vertex of degree 1. $G$ has vertices $\{0, 1\}^n$ and edges represented by 2 boolean circuits $S, P$.

END OF LINE characterizes $\text{PPAD}$; END OF LINE $\leq_p \text{NASH}$ thus NASH is $\text{PPAD}$-hard.

where the output is the unique vertex you get by “following the line” from the given source.

OEOTL is $\text{PSPACE}$-complete!
You are given a node with degree 1 (colored red here)
The highlighted nodes are **PPAD**-complete to find...
The one attached to the red node is \textbf{PSPACE}-complete to find!
Reduce from graph search to NE computation

Use ideas from (DGP’06, CDT’06) that showed the PPAD-completeness

- Intermediate step: search for a panchromatic point of a discrete Brouwer function — in 2D, this a function $f : \mathbb{N} \times \mathbb{N}' \rightarrow \{0, 1, 2\}$ where
  - the bottom row has color 1 (e.g. red)
  - the left-hard side has color 2 (e.g. green)
  - the top and RHS have color 0 (e.g. blue)
  - internal points colored by a poly-size boolean circuit

Assume $N$ and $N'$ are exponentially large
discrete Brouwer function

Search for trichromatic point (here, in 2D)
Search for trichromatic point... they are PPAD-complete to find (Chen and Deng ('06, '09): 2D-Brouwer)
Specific one guaranteed by the line is **PSPACE-complete** to find!
The reduction in a bit more detail

Chen and Deng ('06, '09): 2D-Brouwer
search for trichromatic point

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The crossover gadget
In 3D, no crossover gadget needed; a red-yellow-green line simulates the end-of-line graph.
Discrete Brouwer functions in 3D
Discrete Brouwer functions in 3D
from discrete to continuous Brouwer functions

To continue to games, make a continuous Brouwer function that encodes the discrete one; arrows show directions
Arithmetic circuits can compute a continuous version
Homotopy: use this “basic function” as starting-point
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We have found a \textbf{PSPACE}-complete homotopy! It remains to show that it can be simulated via standard homotopy algorithms for games.
Reduction to LINEAR TRACING; gadgets maintain close correspondence between paths in END OF LINE graph and homotopy paths.

Lemke-Howson, by contrast: $2n$ distinct homotopy paths are allowed (corresp to which strategy to give the large bonus; or the intially dropped label), \textbf{PSPACE}-completeness holds for search for any of those initial choices.
Conclusions and further work

Our result is of the form: here is the complexity of computing the output of this specified (exponential-time) algorithm... what about other algorithms?

Open Problem:

What is the complexity of

\[ n \times n \text{ game } G \]

QUESTION: Does Fictitious play converge for \( G \)? (and if so, give the resulting equilibrium)

Generalizations? Can it be shown \( \text{PSPACE} \)-complete to compute the output of members of a bigger class of algorithms? (not for support enumeration; looks like \( \text{LEXMINSAT} \); \( \text{OptP} \subseteq \text{FP} \subseteq \text{NP} \))

...how about “path-following” algorithms?
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...how about “path-following” algorithms?
Fictitious Play

How it works

- Each player chooses one pure strategy at step 0
- At each step $t$, consider the other player’s sequence of $t$ strategies: make a best response to the uniform distribution over that sequence, add that to your own sequence

Key points: If it converges, it converges to NE. And, it converges for zero-sum games (Robinson 1951), and $2 \times n$ games, but not all $3 \times 3$ games (Shapley 1964).