COMPLEX-VALUED HOUGH TRANSFORMS FOR CIRCLES

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ABSTRACT

This paper proposes the use of complex variables to represent votes in the Hough transform for circle detection. Replacing the positive numbers classically used in the parameter space of the Hough transforms by complex numbers allows cancellation effects when adding up the votes. Cancellation and the computation of shape likelihood via a complex number's magnitude square lead to more robust solutions than the "classic" algorithms, as shown by computational experiments on synthetic and real datasets. We note a resemblance to methods used in quantum theory.

1. INTRODUCTION

Circles are a common geometric structure of interest in computer vision applications, and the Hough transform is one of the main methods used to locate such shapes. The Hough transform is built around a voting scheme where image elements vote for parameters of a geometric object.

The intuition to use complex numbers to replace votes comes from quantum mechanics, where wave functions are modeled using complex numbers, which allow for probabilities (the magnitude square of the wave function) not only to add, but also to cancel out.

To illustrates the cancellation effect consider the sum of a random collection of complex numbers of magnitude one. It is smaller than the sum of random real numbers of magnitude one:

$$\left|\sum_{k=0}^{N-1} e^{i\eta_k}\right| = O(\sqrt{N}) \qquad \text{versus} \qquad \sum_{k=0}^{N-1} |e^{i\eta_k}| = N \ .$$

That is, "noisy" events end up with a small number of votes compared with coherent (equi-phase) events. A similar observation is that the sum of periodic complex numbers of magnitude one annihilates, i.e.,

$$\left|\sum_{k=0}^{N-1} e^{i2\pi k/N}\right| = 0 \qquad \text{versus} \qquad \sum_{k=0}^{N-1} |e^{i2\pi k/N}| = N \ .$$

1.1. Our Contribution

The Hough transform [1, 2] accumulates shape likelihoods in a parameter space, based on points or point+gradient information. The shape is recognized when there are enough votes. Classically, votes do not cancel, but here we will illustrate how the complex number approach leads to cancellation and more robust shape detection[1].

An important technicality in modeling AI systems is the introduction of non-linearities. A notorious recent example is the use of "pooling" in hierarchical, deep learning [4, 5, 6, 7]. In quantum physics, nonlinearities appear in a specific way, via "collapsing the wave function." The probabilities obtained are the squared magnitudes of the wave function. This magnitude operator is a non-linear local operator, the same used in [8, 7].

The use of complex numbers to model likelihoods and of likelihood computation via their magnitude squares emerged from the field of quantum physics. Here, in a similar way, we present the complex number inference as a way to represent and combine information that allows for probability cancellation.

In this paper we modify a voting scheme (the Hough transform for circle detection using tangent information and radius range) by introducing complex-valued likelihoods, and show that the modified version performs better than the classic counterpart. Note that while the focus here is the "Quantum" Hough transform for circles where a thorough study is conducted, the novel method proposed can be considered for other voting schemes.

1.2. Previous Work

Recent related works that are based on complex numbers in a quantum framework are Quantum Signal Processing [9] and Quantum Neural Networks [10]. [9] focused on quantum measurements, consistency, and quantization, but did not address the subject of probability cancellation due to wave in-

[1]In learning theory, the problem of how to combine classifiers – mixture of experts [3] – can be cast as the choice of a voting scheme. We argue that when experts produce complex numbers, as opposed to probabilities, the additive combination of experts is more efficient.

terference or its role to achieve robust inference. In terms of demonstration or experiments, they only studied a simple least-squares error problem – there is no concrete suggestion of the use of complex-numbered probabilities for image analysis. [10] showed how a quantum neural network could compute entanglement, but did not suggest a new inference mechanism based on it.

2. COMPLEX-NUMBER VOTING

The Hough transform is a voting scheme: it accumulates votes in the parameter space of the possible geometrical shapes, and the shapes with enough votes are recognized. It is known to be resistant to outliers. However, in this procedure, there is no cancellation as all votes are added [2].

Here we introduce a complex-number based voting scheme, where relevant pixels produce complex valued, *oscillating* votes. Adding the complex votes from the various sources and then taking the magnitude squared of the sum, allows cancellation of probabilities, which makes it a more robust procedure.

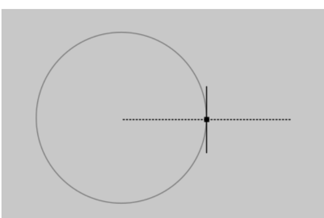


Fig. 1. When tangent information is available, and the radius is unknown, every point votes for the line perpendicular to the tangent.

Often circular points are found together with their orientation so that the voting is only on the line connecting the point to the center of the circle, which is orthogonal to the local edge (Figure 1). If the radii of circles to be found are unknown, a point will vote along the entire line defined by the tangent (and the point).

We study four (4) versions of Hough transforms in this setting, depending if we use or not the tangent magnitude weight (at the point source) and if the value of the vote is a constant (real-numbered) or phase-changing complex numbers:

- Constant votes, weighted sources.
- Constant votes, non-weighted sources.
- Complex votes, weighted sources.
- Complex votes, non-weighted sources.

We detect edges using Morlet wavelets, filtering the image with a bank of wavelets of varying orientations, and fixed scale (varying the scale could be considered, but for our applications the edges are well captured by this one scale). At every point, we look for the wavelet of maximum magnitude $V_{j_{x'}}(x') = \max_j V_j(x')$ (j is the orientation index), and use only the points x' such that $V_{j_{x'}}(x') > \Gamma$ (an empirically defined threshold).

Let $m(x') = V_{j_{x'}}(x')$, and $\tau_{x'}$ be the unit vector perpendicular to the direction of $V_{j_{x'}}$. In the classic Hough transform for circle detection using tangents, every point x on the line $l(x') := x' + \alpha\tau_{x'}$ gets a vote $m^2(x')$. In the complex number case, the vote is $m(x')\, e^{i\nu|x-x'|}$: the magnitude squared is the same as in the classic Hough transform, but a point on the line gets a vote with a phase proportional to the distance to the source. The frequency parameter ν relates to a very rough estimation of the size of the object to be found (radii of the circle) and noise characteristics. As we show in Figure 3 the system is robust across a large range of ν .

The total vote $\phi(x)$ at location x, from all wavelet responses (sources), is shown in the following table:

vote type	*weighted*	$\phi(x)$
constant real	yes	$\sum_{x':x\in l(x')} m(x')$
constant real	no	$\sum_{x':x\in l(x')} 1$
complex	yes	$\sum_{x':x\in l(x')} m(x')e^{i\nu\|x-x'\|}$
complex	no	$\sum_{x':x\in l(x')} e^{i\nu\|x-x'\|}$

The *shape likelihood*, L, is defined as $L(x) = |\phi(x)|^2$, and detecting shapes consists of finding local maxima in L. Figure 2 shows some examples of accumulator spaces under these 4 equations, for 4 types of input images. Notice how the magnitude squared of complex number votes provide much sharper accumulator spaces.

3. EXPERIMENTS

How sensitive are the complex number vote methods to the frequency parameter ν? Empirically, we found that $\nu = \pi/r$, for r the estimated radius, works well, so we made a study in which we vary ν from 10 times smaller to 10 times larger this value, i.e., we studied $\nu = f\pi/r$, for $f = 10^i, i = \{-1, -0.8, ..., 1\}$, to analyze the influence of this parameter.

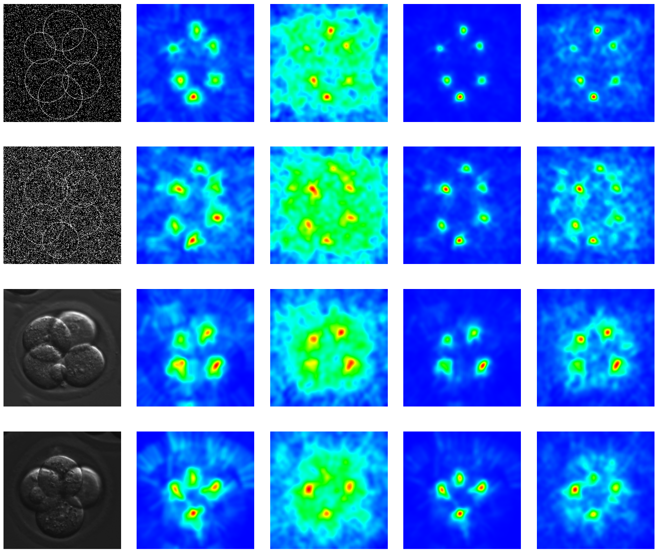
We computed precision and recall values on two sets of 100 synthetic images containing 6 overlapping circles each, under different levels of noise (see examples in Figure 2, top two rows). Figure 3 shows results[3] Notice that the complex number vote methods outperform the classic counterpart regardless of the source weights being considered or not.

Another parameter that affects accuracy is the threshold above which local maxima are found in the accumulator space[4]. To study this parameter, with fixed $f = 1$ (a reason-

[2] As a technique it can also be viewed as mixture of experts (the sources) via additive probabilities, where each vote is a contribution from an expert (a source).

[3] Detection is considered correct (true positive, *tp*) if the located center is at most Γ pixels away from the ground-truth center. If $n > 2$ centers are located at a distance $< \Gamma$ from a ground truth center, $n-1$ are considered false positives, *fp*. False positives are also centers located more than Γ pixels away from any ground truth center. False negatives, *fn*, correspond to lack of detection at a distance $< \Gamma$ from a true center. Precision = tp/(tp+fp). Recall = tp/(tp+fn). We use $\Gamma = 5$ for the experiments on synthetic images, and $\Gamma = 12$ for the experiments on images of cells.

[4] For the synthetic images, we fixed this as 0.1 (of the maximum value).

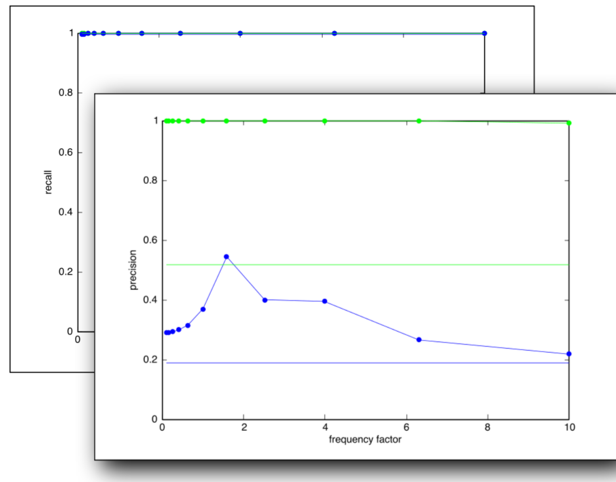


(a) (b) (c) (d) (e)

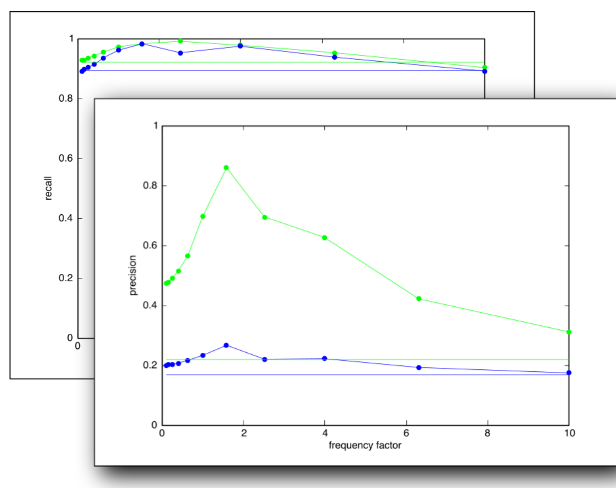
Fig. 2. Outputs of the 4 different methods. (a) Inputs. From top to bottom: synthetic image with noise 0.1, synthetic image with noise 0.2, 4 cells with small degree of overlap, 4 cells with large degree of overlap. (b) Voting space of classic method, weighted by edge strength. (c) Classic, non-weighted. (d) Complex, weighted by edge strength. (e) Complex, non-weighted. **The accumulator spaces for the complex methods are much sharper than the classic counterparts.** Before finding local maxima, in all cases the accumulator spaces are convolved by a gaussian of standard deviation $2\pi r/n$, where r is the estimated radius, and n is the number of wavelet orientations (we use $n = 32$).

able value according to the previous study), we performed experiments on two datasets of 100 images of 4 cells each (from [11]), where the datasets differ in level of cell overlap (examples are shown in Figure 2, bottom two rows). This dataset presents, besides tangent-capture noise, “shape” noise, meaning the cells are not perfect circles, so the center location is more challenging. Figure 4 displays the obtained precision/recall values. Notice that the complex number methods again outperform the classic counterparts. But in this case, the version that works best is the one in which source weights are not considered. This is explained by the fact that, due to illumination conditions, not all tangents belonging to the circle edges are weighted the same, and so it is better to ignore such weights, and rely only on complex number interference, when accumulating votes.

Note on Computational Complexity. While there is more computations in the quantum method due to the use of complex numbers, this increase affects the running time by less than a factor of 2. Moreover, all the algorithms can be implemented in parallel if speed is a concern.

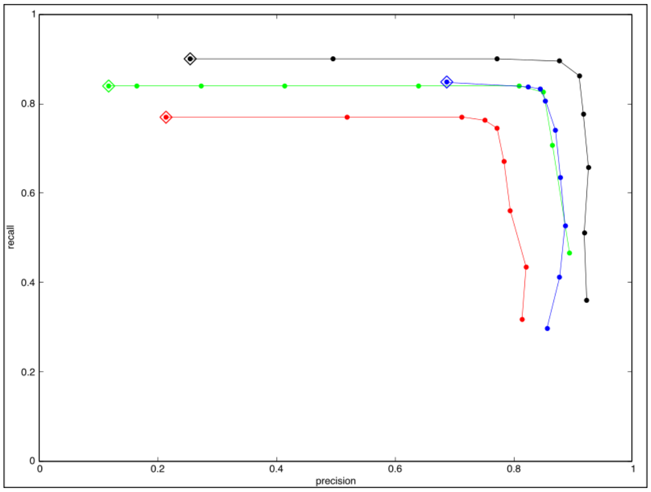


(a)

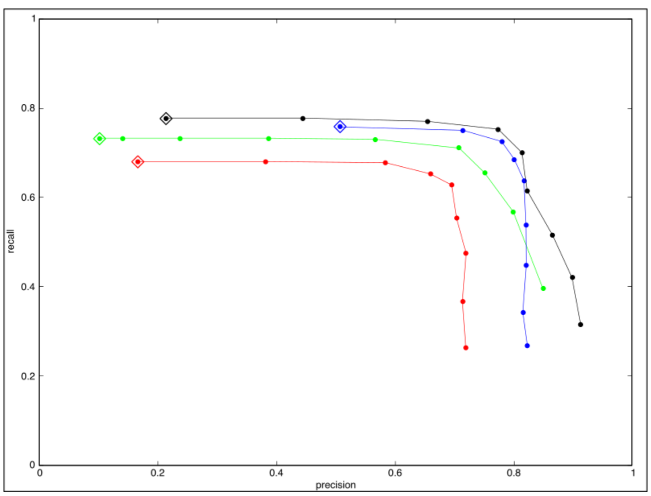


(b)

Fig. 3. Precision (foreground graphs) and recall (background graphs) as a function of the frequency factor on experiments with 100 synthetic images of 6 overlapping circles each, with different levels of noise: 0.1 in (a), and 0.2 in (b) – see Figure 2 (noise α means a pixel value s is changed to $1 - s$ with probability α). If the frequency factor is f, the actual frequency is $f\pi/r$, where r is the estimated radius (in this case 50). Doted, piece-wise linear lines: complex number votes. Horizontal lines: constant votes. Green: source weight considered. Blue: source weight not considered. **These results show that, for the mentioned dataset, the complex number vote methods outperform the constant-vote methods across a wide range of frequency parameters.**



(a)



(b)

Fig. 4. Precision/Recall curves as the threshold for local maxima detection varies, from $1/10$ to $9/10$ in intervals of $1/10$. Dots enclosed by a diamond correspond to the value $1/10$. Red: constant votes, weighted sources. Green: constant votes, non-weighted sources. Blue: complex votes, weighted sources. Black: complex votes, non-weighted sources. (a) Plot of experiment on 100 images of 4 cells each, with small degree of overlap. (b) Equivalent database as in (a) but with larger degree of overlap. Frequency was constant for the complex number vote methods (equal to $\pi/60$ – the estimated cell radius is 60). **The complex vote methods display a better precision/recall trade-off, especially when overlap is high.** The used datasets are publicly available [11].

4. CONCLUSION

We introduced the use of complex numbers and their magnitudes for voting in the Hough transform for circle detection when tangent information is available and radius is not precisely known. We demonstrated that it achieves better performance than the classic (constant, real numbered) counterpart.

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5. REFERENCES

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