1.4.1 Poisson Formula

$$\sum_{n=-\infty}^{\infty} e^{-inT\omega} = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta\Big(\omega - \frac{2\pi n}{T}\Big).$$

Proof: We'll denote the r.h.s. by $c(t)$ and notice that $\hat{c}(\omega)$ is a periodic function with a period of $2\pi/T$. Meaning that it will be enough to show that $\hat{c}(\omega) = \frac{2\pi}{T}\delta(\omega)$ for $\omega \in [-\pi/T, \pi/T]$. We'll use a test-function $phi(\omega)$ to show that

$$\langle \hat{c}, \hat{\phi}\rangle = \lim_{N\to\infty}\int_{-\infty}^{\infty}\sum_{n=-N}^{N} e^{-i\omega nT}\hat{\phi}(\omega)d\omega = \frac{2\pi}{T}\hat{\phi}(\omega)(0).$$

The finite geometric sum can be written as

$$\sum_{n=-N}^{N} e^{-i\omega nT} = e^{i\omega NT}\sum_{n=0}^{2N} e^{-i\omega nT} = \frac{e^{i\omega(N+1/2)T}}{e^{i\omega T/2}}\frac{1-e^{-i\omega(2N+1)T}}{1-e^{-i\omega T}} = \frac{e^{i\omega(N+1/2)T}-e^{-i\omega(N+1/2)T}}{e^{i\omega T/2}-e^{-i\omega T/2}} = \frac{\sin\big((N+1/2)\omega T\big)}{\sin\big(\omega T/2\big)}.$$

($\sum_{i=0}^{n-1} a^i = \frac{1-a^n}{1-a}$). Thus,

$$\langle \hat{c}, \hat{\phi}\rangle = \lim_{N\to\infty}\frac{2\pi}{T}\int_{-\pi/T}^{\pi/T}\frac{\sin\big((N+1/2)\omega T\big)}{\pi\omega}\frac{\omega T/2}{\sin\big(\omega T/2\big)}\hat{\phi}(\omega)d\omega.$$

(the limited integration domain results from ϕ's limited support). We define in $[-\pi/T, \pi/T]$

$$\hat{\psi}(\omega) = \hat{\phi}(\omega)\frac{\omega T/2}{\sin\big(\omega T/2\big)},$$

use the identity that $\int_{-T}^{T} e^{-iwt}dt = \frac{2\sin(T\omega)}{\omega}$, and get from Parseval

$$\langle \hat{c}, \hat{\phi}\rangle = \lim_{N\to\infty}\frac{2\pi}{T}\int_{-\infty}^{\infty}\frac{\sin\big((N+1/2)\omega T\big)}{\pi\omega}\hat{\psi}(\omega)d\omega = \lim_{N\to\infty}\frac{2\pi}{T}\int_{-(N+1/2)T}^{(N+1/2)T}\psi(t)dt.$$

(note that the 2π falls from the 2 in the sinc and π from the denominator, and replaced by the same from the Parseval) When n goes to infinity the integral equals to $\hat{\psi}(0)$ which, in turn, equals to $\hat{\phi}(0)$.