Combinatorial Auctions with Decreasing Marginal Utilities[1] [2]

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In most of microeconomic theory, consumers are assumed to exhibit decreasing marginal utilities. This paper considers combinatorial auctions among such *submodular* buyers. The valuations of such buyers are placed within a hierarchy of valuations that exhibit no complementarities, a hierarchy that includes also OR and XOR combinations of singleton valuations, and valuations satisfying the gross substitutes property. Those last valuations are shown to form a zero-measure subset of the submodular valuations that have positive measure. While we show that the allocation problem among submodular valuations is NP-hard, we present an efficient greedy 2-approximation algorithm for this case and generalize it to the case of limited complementarities. No such approximation algorithm exists in a setting allowing for arbitrary complementarities. Some results about strategic aspects of combinatorial auctions among players with decreasing marginal utilities are also presented.

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1. INTRODUCTION AND OVERVIEW

1.1. Background

Recent years have seen a surge of interest in combinatorial (also called combinational) auctions, in which a number of non-identical items are sold concurrently and bidders express preferences about combinations of items and not just about single items. Thus, for example, a bidder may offer \$40 for the combination of a Tournedos Rossini and a bottle of Chateau Lafitte, but offer only \$10 for each of those items alone. Similarly, a bidder may make an offer of \$10 for a blue and for a red shirt, but not be willing to pay more than \$10 even if he gets both shirts. In general, a combinatorial auction allows bidders to express complementarities – where the value of a combination of packages of items is worth more than the sum of the values of the separate packages – and substitutabilities – where the value of a combination of packages is less than the sum of the values of the separate packages. Such combinatorial auctions have been suggested for a host of auction situations such as those for spectrum licenses, pollution permits, landing slots, computational resources, and others. See de Vries and Vohra(forthcoming) for a survey.

Implementation of combinatorial auctions faces several challenges including the representational question of succinctly specifying the values of the different packages, the algorithmic challenge of efficiently solving the resulting, NP-hard, allocation problem, and the game-theoretic questions of bidders' strategies. These questions have been recently approached by a host of researchers both in the general case and in several interesting special cases: MacKie-Mason and Varian(1994), Varian(1995), Rothkopf and al.(1998), Sandholm(1999), Sandholm and Suri(1999), Fujishima and al.(1999), Tennenholtz(2002), Lehmann and al.(forthcoming), Gonen and Lehmann(2000), and Nisan(2000).

Somewhat surprisingly, the special case that is most often studied in economics has received very little attention from the computational point of view. In most of microeconomic theory, consumers are assumed to exhibit diminishing marginal utilities. In particular, such consumers exhibit *no complementarities*. While it is not clear to what extent this assumption may be justified in practice, it is widely used. At least, it is mathematically convenient, but we think it is often justified in practice. Many papers dealing with computational issues in combinatorial auctions focused on the dual case of no substitutes, i.e., much computational research assumes that a buyer places bids for packages of items and is not interested in a sub-package. In contrast, economists who dealt with auctions did mostly consider auctions in which players expressed no complementarities. For example, for multi-unit auctions, Vickrey's seminal paper Vickrey(1961) assumes *downward sloping* valuations for buyers. Recent papers dealing with combinatorial auctions such as Kelso and Crawford(1982), Gul and Stac-

chetti(1999), Gul and Stacchetti(2000), Ausubel(2000) and Milgrom(2000) usually assume the *gross substitutes* property. Each of these notions implies lack of complementarities.

Perhaps this lack of computational attention is due to a representational difficulty: it is usually assumed that buyers express their preferences by non-exclusive bids on sets of items. Representing buyers with no complementarities seems to require the use of exclusive bids for sets of items (XOR-bids), and the mechanics of such exclusive bids have never been studied in full. Bids on packages of items inherently express a very strong form of complementarity: sub-packages are not valued at all, and therefore such bids seem to provide a very poor starting point to describe types that exhibit no complementarities. The prevalent method for dealing with substitutabilities is the *phantom good* approach of Fujishima and al.(1999) and Nisan(2000), that simulates exclusive bids by non-exclusive bids on an enlarged set of goods. This is certainly not a promising approach for studying the specifics of exclusive bids or types exhibiting no complementarities.

In this paper, we study the three notions found in the literature for lack of complementarities. From the most general to the most specific these notions are, informally:

- **No Complementarities**: The value of a combination of bundles is no more than the sum of the bundle values.
- **Decreasing Marginal Utilities**: The marginal value of an item decreases as the set of items already acquired increases.
- **Gross Substitutes**: The demand for an item does not decrease when the prices of other items increase.

This paper's main message is that the case of *decreasing marginal utilities*, which is equivalent to submodularity of the valuation functions, should be the focus of interest.

We prove three types of results:

- **Representation and Hierarchy:** We study the question of how to represent valuations (i.e., bids) that exhibit no complementarities. Our representations are constructed out of atomic valuations that offer a price for a single item, (*singleton valuations*), using the operations of OR and XOR. Using restricted classes of such bidding languages we are able to obtain a structural hierarchy of families of complement-free valuations. Those *syntactic* classes are compared to three *semantic* classes. This hierarchy contains five strictly nested classes of valuations: (a) complement-free valuations; (b) XOR-of-OR-of-singleton valuations; (c) submodular valuations (d) valuations that satisfy the gross substitutes property; and (e) OR-of-XOR-of-singleton valuations. We prove the containment of each class within the previous one and exhibit examples showing that the containment

is strict. We also prove theorems regarding closure properties of these classes under the operations of OR and XOR. We finally show that in terms of measure the significant gap is between the class of submodular valuations and the class of valuations that satisfy the gross substitutes condition.

- **Allocation:** We then focus our attention on the algorithmic question of optimal allocation among players with submodular valuations. We stress the difference between this case and the more restricted and well-studied case of allocation among valuations satisfying the gross substitutes property. In the latter case, it is known that a Walrasian equilibrium exists by a result of Kelso and Crawford(1982), while in the former Walrasian equilibria do not necessarily exist: see Gul and Stacchetti (1999). Walrasian equilibria may be found in polynomial time and are the basis of almost all known computationally efficient allocation algorithms. One may thus say that the submodular case is the first hard case from a computational point of view[3].

  We first show that the optimal allocation problem remains NP-hard even among agents of submodular types. Our main positive algorithmic result is a simple greedy algorithm that produces an allocation that is a 2-approximation, i.e., an allocation whose value is at least half of the optimal one. This result is then generalized to types exhibiting marginal utilities that may increase only in a bounded manner. This is in sharp contrast to the general case where it remains NP-hard to find even a $n^{1/2-\epsilon}$-approximation: Sandholm(1999) and independently Lehmann and al.(forthcoming).

  We do not know if a better approximation ratio is possible, nor do we know whether a polynomial time approximation algorithm exists that works for all complement-free valuations.

- **False Name Bids:** Our next concern is with the strategic issue of *false-name bids* that was identified and analyzed in a sequence of papers: Sakurai and al.(1999), Yokoo and al.(2000a, 2000b, 2001). This concerns the following problem that is inherent in combinatorial auctions that use the Vickrey-Clark-Groves payment rules (see MacKie-Mason and Varian(1994)) – the only choice, if one requires incentive compatibility and efficiency. Sakurai and al.(1999) observed that a bidder can manipulate a VCG combinatorial auction and reduce his payment by splitting his bid and placing two separate bids under *false-names*. They showed that, in the case of single-item multi-unit

[3] A similar phenomena exists from the information transfer point of view: any optimal allocation algorithm among submodular valuations requires exponential communication, while if the valuations satisfy the gross substitutes property then polynomial communication suffices by a result of Nisan and Segal(2001).

auctions, this type of savings cannot occur when all bids are downward sloping. It is claimed there (without proof and without precise definitions) that this generalizes to combinatorial auctions when all bidders have no complementarities. We show that this is not exactly the case[4], but rather the generalization to combinatorial auctions requires that the *combined valuation* of other bidders is *submodular*. We provide an example showing that it is not sufficient to require that each player be of a submodular type.

1.2. Paper Structure

This paper is charting the so far unexplored territory of complement-free valuations. It is therefore by necessity proceeding carefully and proving a large number of small results. In section 2 we present the basic definitions of the classes of valuations that we discuss. Section 3 discusses the representation of valuations using OR and XOR expressions, and provides a structural hierarchy of sub-classes of complement-free valuations. It also shows that the gap between submodular and gross substitutes valuations is very large. Section 4 provides many examples of complement-free and submodular valuations, both natural ones. Such examples are used to separate the different classes defined and studied previously. Section 5 presents the main result of this paper. It discusses allocation (winner determination) algorithms for submodular agents, comparing with the gross substitutes case. Section 6 discusses false-name bids. Finally, section 7 briefly mentions an extension of the results of section 5 to valuations exhibiting limited complementarities and bounded increases in marginal utilities.

2. DEFINITIONS

2.1. Preliminaries

In this paper we consider a combinatorial auction of non-identical goods. There is a set X of items for sale by a single auctioneer in a single combinatorial auction. We shall denote the number of items for sale by $\mathrm{m} = \mid X \mid$. There are n agents (buyers, bidders) who all desire these items. Each agent i has its own private valuation function, v_i, that specifies the utility it gets from each possible subset of items: i.e. for a subset $A \subseteq X$ of items, $v_i(A)$ is the amount of money at which bidder i values this subset A of items. For a singleton set $\{x\}$ we will use $v_i(x)$ as a shorthand for $v_i(\{x\})$. This formalization makes two important implicit assumptions:

- **Quasi-linearity:** Agents' utilities can be measured in terms of "money", i.e. are linear in the "money".

[4]We have contacted the authors of this paper and they have confirmed a bug in their unpublished proof.

- **No Externalities:** An agent's valuation depends only on the set of items he wins, not on the identity of the agents that get the items outside A: the valuation function v_i is $v_i = v_i(A)$, where A is the set of items won by agent i.

In addition, we assume that these valuations all satisfy the following conditions:

- **Free disposal:** Items have non-negative value. Thus v_i satisfies $v_i(A) \leq v_i(B)$, whenever $A \subseteq B$.
- **Normalization:** $v_i(\emptyset) = 0$.

DEFINITION 1. A *valuation* is a function, $v : 2^X \to \mathcal{R}^+$, from subsets of items to the non-negative reals that satisfies the normalization and free disposal conditions, i.e. $v(\emptyset) = 0$ and $v(A) \leq v(B)$, whenever $A \subseteq B$ (monotonicity).

The auctioneer's aim is to find an optimal *allocation*.

DEFINITION 2. An *allocation* is a partition of X into pairwise disjoint sets of items $S_1 \ldots S_n$ that allocates the bundle S_i to agent i. An allocation is *optimal* if it maximizes $\sum_i v_i(S_i)$.

The auction rules must also define the payments received from each bidder.

2.2. Marginal Valuations

A central notion that we will be using is that of the marginal valuation. It describes how a player would value sets of remaining items if he were already given some other items.

DEFINITION 3. Given a valuation v on a set X of items and a set $W \subseteq X$ of items, the *marginal valuation* of a set $A \subseteq X - W$ given W is defined by:

$$v(A \mid W) = v(A \cup W) - v(W).$$

One may consider the marginal valuation of a single element, $v(x \mid \cdot)$, as a discrete analog for the partial derivative of v in the direction x.

2.3. No Complementarities

DEFINITION 4. A valuation v is called *complement-free* if it satisfies:

$$v(A) + v(B) \geq v(A \cup B)$$

for all sets A, $B \subseteq X$. The class CF is the set of all complement free valuations.

While this notion is clearly natural, it turns out that valuations with this property can still have *hidden complementarities*. Example 4 shows that even if v is in CF, the valuation $v(\cdot \mid W)$ need not be, i.e., once a set of items W is already acquired, complementarities may surface. Indeed, as Example 4 shows, this condition of complement-freedom is not the proper analog of downward-sloping valuations. It is therefore only natural to turn to those valuations that have no such hidden complementarities: those valuations whose marginal valuations exhibit no complementarities. It turns out that these are exactly the submodular valuations, i.e., valuations with decreasing marginal utilities.

2.4. Decreasing Marginal Utilities – Submodular Valuations

This paper focuses on a subset of valuations that is of special economic interest: valuations with decreasing marginal utilities. These valuations are a strict subset of the complement-free valuations. We believe that, in fact, it is the *correct* economic notion capturing *absolute lack of complementarity* and that it is a much more natural assumption than the *convexity of preferences* that is so central to much of economic theory. Such valuations are known as submodular functions. In many combinatorial problems, submodularity can play a role similar to convexity in optimization.

DEFINITION 5. A valuation v is called *submodular* if for every two sets of items $S \subseteq T$ and element x, $x \notin T$, $v(x \mid T) \leq v(x \mid S)$. Submodular valuations are also called valuations with *decreasing marginal utilities*. The class SM is the set of all submodular valuations.

Thus we require that the marginal utility of an element decreases as the set of items already acquired increases. Submodular functions are well-known and heavily used in combinatorics (see Narayanan(1997)). If we extend the analogy between the marginal valuation and the derivative discussed in section 2.2, we see why submodular valuations are considered a discrete analog of convex functions – the "derivative" $v(x \mid \cdot)$ is decreasing. Here, we consider only valuations: monotone, positive submodular functions. In the literature, submodular functions have been generally been considered in a wider setting. Many equivalent characterizations of decreasing marginal utilities are well-known.

THEOREM 1 (see Moulin(1988) or Narayanan(1997)). *A valuation v is submodular if and only if any one of the following equivalent propositions holds.*

- *For any $x, y \in X$ and $S \subseteq X$: $v(x \mid S) \geq v(x \mid S \cup \{y\})$.*
- *For any $S, T, V \subseteq X$, such that $S \subseteq T$: $v(V \mid S) \geq v(V \mid T)$.*
- *For any $A, B \subseteq X$: $v(A) + v(B) \geq v(A \cup B) + v(A \cap B)$.*

From the last characterization we get:

COROLLARY 1. *A submodular valuation is complement-free.*

The converse is not true and in Example 4 below we exhibit a valuation that is complement-free, yet its marginal utilities are not decreasing. It turns out that valuations with decreasing marginal utilities are exactly the valuations without any "hidden" complementarities.

LEMMA 1. *[see Topkis(1998)] A valuation v is submodular if and only if for every subset of items R, the marginal valuation function $v(\cdot \mid R)$ is complement-free.*

In particular we get the corollary.

COROLLARY 2. *If a valuation v is submodular, then for every subset R of items the marginal valuation function $v(\cdot \mid R)$ is submodular.*

Proof. All the marginal valuation functions of $v(\cdot \mid R)$ are also marginal valuation functions of v and thus have no complementarities. ∎

2.5. Gross Substitutes

All the definitions and theorems in this subsection are from Kelso and Crawford(1982), Gul and Stacchetti(1999). We have sometimes slightly modified the terminology and notation. We have kept with the now traditional terminology of *gross substitutes*, even though *gross* does not fit the situation. In order to define the gross substitutes property we need to consider first the effect of putting prices on items. We think of what happens to the valuation of v of a set S, when the price of all items in S must be paid.

DEFINITION 6. Given a vector of real item prices $\vec{p} = (p_1 \dots p_m)$, the *surplus* of a set of items S relative to these prices is defined as $v(S \mid \vec{p}) = v(S) - \sum_{i \in S} p_i$. A set S is a *preferred set* of v at prices $\vec{p}$ if $v(S \mid \vec{p}) = \max_T v(T \mid \vec{p})$, i.e., S maximizes the surplus. The *demand set* of v at prices $\vec{p}$ is the set of all preferred sets $D(v \mid \vec{p}) = \{S \mid v(S \mid \vec{p}) = \max_T v(T \mid \vec{p})\}$.

The gross substitutes property mandates that increasing the price of an item will not decrease the demand for any other item: i.e., if an item i is in a preferred set at prices $\vec{p}$, then increasing p_j for some $j \neq i$, will still have item i in some preferred set.

DEFINITION 7. A valuation v is said to satisfy the *gross substitutes* property if for any price vectors $\vec{p}$ and $\vec{q} \geq \vec{p}$ (coordinatewise comparison), if $S = \{j \in X \mid p_j = q_j\}$ and $A \in D(v \mid \vec{p})$, then there exists $A' \in D(v \mid \vec{q})$ such that $S \cap A \subseteq A'$. The class GS is set of all valuations that satisfy the gross substitutes property.

An equivalent condition is called the *single improvement property*. This condition states that a non-preferred set can always be improved (in terms of its surplus) by deleting at most one element from it and inserting at most one element into it.

LEMMA 2 (Gul and Stacchetti(1999)). *A valuation v satisfies the gross substitutes property if and only if for any $\vec{p}$ and $A \notin D(v \mid \vec{p})$, there exists A' such that $\mid A' - A \mid \leq 1$ and $\mid A - A' \mid \leq 1$ and $v(A' \mid \vec{p}) > v(A \mid \vec{p})$.*

The gross substitutes property is stronger than submodularity.

LEMMA 3 (Gul and Stacchetti(1999)). *A valuation that satisfies the gross substitutes property is submodular.*

The key property that makes valuations with the gross substitutes property so convenient is that in an auction (equivalently an exchange economy) with such valuations, Walrasian equilibria exist. A Walrasian equilibrium is a vector of prices on the items and an allocation such that every bidder receives a preferred set at these prices.

DEFINITION 8. A Walrasian equilibrium in an auction with valuations $v_1 \ldots v_n$ is a price vector $\vec{p}$ and an allocation $A_1 \ldots A_n$ such that for all bidders j: $A_j \in D(v_j \mid \vec{p})$.

THEOREM 2 (Kelso and Crawford(1982)). *Any auction with gross substitutes valuations $v_1 \ldots v_n$ has a Walrasian equilibrium.*

It turns out that this is essentially also a necessary condition.

THEOREM 3 (Gul and Stacchetti(1999)). *If v_1 does* not *satisfy the gross substitutes property then there exist valuations $v_2 \ldots v_n$ that do satisfy the gross substitutes property*[5] *and the auction with $v_1 \ldots v_n$ does* not *have a Walrasian equilibrium.*

It is easy to verify the "first Welfare theorem" in this context: any Walrasian equilibrium gives an optimal allocation, i.e. one that maximizes $\sum_j v_j(A_j)$.

3. REPRESENTATION AND STRUCTURE

3.1. Elements of Representation

The question of representation of valuations, i.e., how to succinctly describe a valuation without listing explicitly the values for each of the $2^n - 1$ subsets of items, must be addressed before valuations can be efficiently

[5] In fact, Gul and Stacchetti(1999) prove that $v_2 \ldots v_n$ may be unit demand functions – a stronger claim.

treated in any allocation algorithm. How can we represent complement-free valuations succinctly? We may consider representing such valuations by combining *simple* valuations by suitable operators.

The basic syntactic elements from which valuations are usually constructed consist of a certain price for a specific set of elements. They correspond to the single-minded valuations (i.e., bidders) of Lehmann and al.(forthcoming). The only single-minded valuations in CF are valuations in which only a single item is valued at a positive value. It is therefore expected that those *singleton* valuations will play a central role in describing CF valuations.

DEFINITION 9. For an item $x \in X$ and a price p, the singleton valuation e_x^p is a valuation giving the same value p to all sets containing x and the value 0 to all sets that do not include x.

The operators commonly used for representing valuations are those introduced in Sandholm(1999): OR and XOR. They are often understood as prescriptions concerning the compatibility of bids, but the following presentation, as operations on valuations, is more convenient.

DEFINITION 10 (Nisan(2000)). Let v_1 and v_2 be two valuations on the set X of items. The valuations $v_1 \oplus v_2$ (XOR) and the valuation $v_1 \vee v_2$ (OR) are defined by:

$$(v_1 \oplus v_2)(S) = \max(v_1(S), v_2(S)),$$

$$(v_1 \vee v_2)(S) = \max_{T \subseteq S} (v_1(T) + v_2(S - T)).$$

Both operators have an intuitive description. The OR ($\vee$) of two valuations represents the valuation of an agent representing both valuations, bidding on their behalf and sharing the result between them optimally. The value given to a bundle by the OR of two valuations is obtained by partitioning the bundle optimally between these valuations. The XOR ($\oplus$) of two valuations represents the valuation of a single bidder capable of choosing between two possible but incompatible personalities after the auction. His value for a set S is the largest of the two values of the component valuations. The operations OR and XOR are obviously associative and commutative.

3.2. A Syntactic Hierarchy

It is now natural to study the hierarchy of valuations obtained by combining singleton valuations by the OR and XOR operations. Here are the levels of the hierarchy

1. Singleton valuations.

2. *$OS-valuations$*: The family of valuations that can be described by OR of singletons is precisely the family of (separately) *additive valuations*: those valuations which value any set at the sum of the values of its elements. They exhibit no complementarities and no substitutabilities.

3. *$XS-valuations$*: The family of valuations that can be described by XOR of singletons is usually called the family of *unit-demand valuations* Gul and Stacchetti(1999): those valuations that value any set of elements at the value of the element of the set that is most valued. This family of types has been considered central by Vickrey and since.

4. *$OXS-valuations$*: The family of OR's of XS valuations represents the aggregated valuation of multiple unit-demand players.

5. *$XOS-valuations$*: The family of XOR's of additive valuations turns out to subsume the whole hierarchy. This follows from the simple observation that OR distributes over XOR:

$$(a \oplus b) \vee (c \oplus d) = (a \vee c) \oplus (a \vee d) \oplus (b \vee c) \oplus (b \vee d).$$

Therefore XOS is closed under OR (it is obviously closed under XOR). It follows that any expression that uses combinations of OR and XOR operations on singleton valuations can be represented as a simple XOS expression.

We shall see in Section 4.2 that XOS is a strict subset of CF, but contains SM and that OXS is a strict subset of XOS. In fact we have just shown that on any set of atomic valuations (not only singletons), XOR-of-OR combinations can represent arbitrary complex combinations of ORs and XORs.

3.3. Closure Theorems

Once we have defined the operations of OR and XOR it is natural to ask whether the classes of valuations defined semantically (CF, SM, and GS) are closed under these operations.

3.3.1. CF

The operations of OR and XOR fit well our concern with the class CF since:

THEOREM 4. *The class CF is closed under OR and XOR.*

Proof. Let v_1 and v_2 be complement free valuations then:

$$(v_1 \oplus v_2)(S \cup T) = max(v_1(S \cup T), v_2(S \cup T)) \leq$$

$$max(v_1(S) + v_1(T), v_2(S) + v_2(T)) \leq$$
$$max(v_1(S), v_2(S)) + max(v_1(T), v_2(T)) =$$
$$(v_1 \oplus v_2)(S) + (v_1 \oplus v_2)(T).$$

Similarly fix partitions $(S_1 : S_2)$ of S and $(T_1 : T_2)$ of T, such that $(v_1 \vee v_2)(S \cup T) = v_1(S_1 \cup T_1) + v_2(S_2 \cup T_2)$. Now,

$$(v_1 \vee v_2)(S \cup T) = v_1(S_1 \cup T_1) + v_2(S_2 \cup T_2) \leq$$
$$v_1(S_1) + v_1(T_1) + v_2(S_2) + v_2(T_2)) \leq$$
$$(v_1 \vee v_2)(S) + (v_1 \vee v_2)(T).$$

∎

3.3.2. SM

The class of submodular valuations is *not* closed under either OR or XOR. It is not closed under XOR as it contains the class OS of additive valuations, but Example 3 shows that it does not contain the class XOS which is XORs of additive valuations. The fact that it is not closed under OR is shown by Example 5.

3.3.3. GS

The class GS is not closed under XOR as it contains the class OS of additive valuations, but Example 3 shows that even its superset SM does not contain the class XOS which is XORs of additive valuations. It is however closed under the OR operation, as will be shown now.

THEOREM 5. *If valuations u and w satisfy the gross substitutes property then so does $u \vee w$.*

The theorem follows from combining the results of Fujishige and Yang(2001) showing that gross substitutes is equivalent to $M^\# - concavity$ with those of Murota and Shioura(1999) regarding the closure of $M^\#$-convex functions under convolution (i.e., OR). For completeness, we provide a simple proof based on the results cited above of Kelso and Crawford(1982) and Gul and Stacchetti(1999).

Proof. We will prove the theorem indirectly, applying Theorems 3 and 2. Using Theorem 3, it suffices to show that for all gross substitutes valuations $v_2 \ldots v_n$, an auction among agents $u \vee w, v_2 \ldots v_n$ has a Walrasian equilibrium. Theorem 2 states that an auction among agents $u, w, v_2, \ldots, v_n$ has a Walrasian equilibrium: vector of prices $\vec{p}$ with allocation $A_u, A_w, A_2, \ldots A_n$. We claim that $\vec{p}$ with the allocation $A_u \cup A_w, A_2, \ldots A_n$ is a Walrasian equilibrium for bidders $u \vee w, v_2 \ldots v_n$.

The only thing that needs to be verified is that $(A_u \cup A_w) \in D(u \vee w \mid \vec{p})$. This is so since for any set S, let S_u and S_w be a partition of S such that $(u \vee w)(S) = u(S_u) + w(S_w)$. Thus also $(u \vee w)(S \mid \vec{p}) = u(S_u \mid \vec{p}) + w(S_w \mid \vec{p})$. Now, since $A_u \in D(u \mid \vec{p})$ we have that $u(S_u \mid \vec{p}) \leq u(A_u \mid \vec{p})$, and similarly for w. Clearly $u(A_u) + w(A_w) \leq (u \vee w)(A_u \cup A_w)$, and so also $u(A_u \mid \vec{p}) + w(A_w \mid \vec{p}) \leq (u \vee w)(A_u \cup A_w \mid \vec{p})$. Thus

$$(u \vee w)(S \mid \vec{p}) = u(S_u \mid \vec{p}) + w(S_w \mid \vec{p}) \leq$$
$$\leq u(A_u \mid \vec{p}) + w(A_w \mid \vec{p}) \leq (u \vee w)(A_u \cup A_w \mid \vec{p}).$$

∎

3.4. The Complete Hierarchy

We have seen by now five significant classes of valuations. Three of them were defined semantically: complement-free (CF), submodular (SM), and gross substitutes (GS). Two of them were defined syntactically: ORs of XORs of singletons (OXS) and XORs of ORs of singletons (XOS). We now can state the relationships between these classes.

THEOREM 6.

$$OXS \subset GS \subset SM \subset XOS \subset CF.$$

All containments are strict.

Proof. We will prove the containments from left to right. The fact that they are strict will follow from the examples found below in Section 4.2.

1. $OXS \subseteq GS$: This follows from Theorem 5 since GS contains all unit demand valuations (XS-valuations) (Gul and Stacchetti(1999)), and therefore also contains ORs of XS valuations.
2. $GS \subseteq SM$: This was shown in Gul and Stacchetti(1999).
3. $SM \subseteq XOS$: each submodular valuation may be represented by a long XOR expression. For each permutation π of the items we will have an OR clause – the XOS expression will be the XOR of all these OR clauses. The OR clause for the permutation π will offer for each item i the marginal price of i assuming that all items preceding it in the permutation π have already been obtained. Formally, for $i = \pi(j)$, the price is $v(i \mid \{\pi(1), \ldots, \pi(j-1)\})$. To see that the XOS expression indeed represents the submodular valuation, consider a submodular v and a set A of items. For any permutation π of the items in which the items of A are placed first, the value given to A by the OR clause for π is exactly $v(A)$. All other OR clauses give to A a value that is smaller or equal, since v has decreasing marginal utilities.
4. $XOS \subseteq CF$: By Theorem 4.

∎

3.5. Measure

In this section we study how *large* are the five classes of valuations studied so far. Our sense of size is the simple one of measure. Each valuation is specified by a vector of $2^m - 1$ real numbers, and we can thus look at each class of valuations as residing in the $2^m - 1$-dimensional Euclidean space. What we ask here is what is the measure of these classes, in particular whether their measure is positive. It turns out that there is a significant gap between the GS and SM levels of the hierarchy: GS valuations have measure 0, while SM valuations have positive measure.

THEOREM 7. *For every number of items $m \geq 3$, the class GS (on m items) has zero-measure in the $2^m - 1$-dimensional Euclidean space. Therefore its interior is empty.*

Proof. The theorem is directly implied by the following claim:

CLAIM 1. *If $x, y, z \in \Omega$ are pairwise distinct items and v is a GS valuation such that $v(z \mid y) < v(z \mid x)$, then $v(x \mid y) = v(x \mid z)$.*

The theorem follows from the claim since the set GS is contained in the union of the three hyper-planes defined by the equations $v(x \mid y) = v(x \mid z)$ for all permutations of x, y, z.

The claim follows from the two lemmas below. Each of those lemmas is proved by a similar technique, making a direct use of a GS valuation: the prices of two of the three items are fixed and the price of the third item is made to increase from zero to a large value.

LEMMA 4. *If $v(z \mid y) < v(z \mid x)$, then $v(x \mid y) \geq v(x \mid z)$.*

Proof. Given a valuation v, a price vector (for all items), and sets A, B of items, we write $A \preceq B$ or $B \succeq A$ if B is at least as preferred as A at the given prices, $A \prec B$ or $B \succ A$ if B is strictly preferred to A and $A \sim B$ if A and B have the same net utility given the prices. Let us fix the prices of x and y: $p_x = v(x \mid z)$ and $p_y = v(y \mid z)$. These prices induce preferences between subsets of Ω. First: $xz \sim z \sim yz$. Notice that this holds independently of the price of z. Notice also that, since v is GS it is submodular, i.e., it exhibits decreasing marginal values and, since one does not gain by adding x to z, one cannot gain by adding x to yz and therefore $xyz \preceq yz$. Let us fix for all items of $\Omega - \{x, y, z\}$ prices that are high enough to ensure none of those items are in any preferred set. Let us now fix, at first, the price of z at zero: $p_z^1 = 0$. One never loses by adding z to one's bundle and therefore all three bundles: z, xz and yz are preferred bundles. In particular xz is a preferred set at the current prices. Let us now increase the price of z to some value, p_z^2 such that $v(z \mid y) < p_z^2 < v(z \mid x)$. At the new prices $yz \prec y$ and $x \prec xz$. Therefore $x \prec y$. By the gross substitutes property, there is, at the new prices, some preferred set that contains x. This set must be at least as preferred as y

and therefore cannot be x and certainly not xz or xyz. Therefore xy is a preferred set and $xy \succeq y$: $v(x \mid y) - p_x \geq 0$ and $v(x \mid y) \geq v(x \mid z)$. ∎

LEMMA 5. *If* $v(z \mid y) < v(z \mid x)$, *then* $v(x \mid y) \leq v(x \mid z)$.

Proof. By contradiction. Assume $v(z \mid y) < v(z \mid x)$ and $v(x \mid y) > v(x \mid z)$ and fix prices for x and z such that: $v(x \mid z) < p_x < v(x \mid y)$ and $v(z \mid y) < p_z < v(z \mid x)$. Those prices for x and z imply the following preferences: $yz \prec y \prec xy$ and $x \prec xz \prec z$. Since v is submodular, we also know that $xyz \prec yz$ (one does not gain by adding x to z, therefore one cannot gain by adding x to yz) and that $\emptyset \prec x$ (one gains by adding x to y, therefore one gains by adding x to the empty bundle). Let us fix for all items of $\Omega - \{x, y, z\}$ prices that are high enough to ensure none of those items are in any preferred set. Let us now fix, first, the price of y at zero. Then, xy is a preferred set. Let, then, the price of y increase to a value high enough to make any bundle containing y unattractive. Then z is the only preferred bundle, contradicting the substitutes property. This concludes the proof of the lemma, of the claim, and of the theorem. ∎

∎

We do not know what is the dimensionality of the class GS. In particular it is possible that its dimension is only polynomial in m. We have shown that the class GS is *small*, let us now show that SM is *large*.

THEOREM 8. *For any number of items* $m \geq 1$, *the class SM of valuations over* m *items has a non-empty interior in the* $2^m - 1$*-dimensional Euclidean space. Therefore it has positive measure.*

Proof. We shall use $\mid x \mid$ to denote the absolute value of a number x and $\mid S \mid$ to denote the cardinality of a finite set S. Let $n = \mid \Omega \mid$. The valuation $v_c(S) = 1 - 2^{-|S|}$ is a symmetric (see section 4.1.1), monotone, normal, submodular valuation for which the marginal value of the kth element is 2^{-k}. It is, in a sense, a *central* submodular valuation. The theorem follows from the claim that, for any function $h : 2^\Omega \to \mathcal{R}$ such that $\mid h(S) \mid \leq 2^{-(n+2)}$, the valuation $v(S) = v_c(S) + h(S)$ is submodular. Indeed, assume $S \subset T$ and $x \notin T$. We have:

$$v(x \mid S) = v(S \cup \{x\}) - v(S) \geq 2^{-(|S|+1)} - 2^{-(n+1)} \geq$$

$$2^{-(|S'|+1)} + 2^{-(|S'|+1)} - 2^{-(n+1)} \geq 2^{-(|S'|+1)} + 2^{-(n+1)} \geq$$

$$v(x \cup S') - v(S) = v(x \mid S').$$

∎

4. EXAMPLES OF SUBMODULAR VALUATIONS

4.1. Natural Examples

The simplest and most common natural examples of valuations with no complementarities are the additive valuations (the class OS) and the unit demand valuations (the class XS) mentioned in Section 3.2. These valuations are both trivially in sub-classes of OXS and thus clearly satisfy the gross substitutes property, and are also submodular. We now mention some other natural valuations that are submodular.

4.1.1. Symmetric valuations

Symmetric valuations are ones where $v(S)$ depends only on the size of S, $\mid S \mid$. These cases fit auctions of multiple identical items. Symmetric valuations may be described by a sequence of *marginal* values: numbers $p_1 \ldots p_m$ where $p_i = v(T) - v(S)$ for any sets T and S of sizes i and $i-1$ respectively. We then have: $v(S) = \sum_i^{|S|} p_i$ for any set S. The following set of symmetric valuations has received considerable attention. For example Vickrey(1961)'s multi-unit auction considers only such valuations.

DEFINITION 11. A symmetric valuation is downward sloping iff for all i, $p_{i+1} \leq p_i$.

As expected, these are exactly the symmetric submodular valuations.

PROPOSITION 1. *A symmetric valuation is downward sloping if and only if it is submodular.*

Proof. For a symmetric valuation v, for $x \notin S$, $v(x \mid S) = p_{|S|+1}$, and the equivalence is immediate. ∎

It is known from Gul and Stacchetti(1999) that downward sloping symmetric valuations also satisfy the gross substitutes property and in fact can be represented in the class OXS (see Nisan(2000)). On the other hand, the complement-free symmetric valuations are a wider class – see below Examples 4 and 3.

4.1.2. Additive valuations with a budget limit

Our second class of examples is composed of valuations that are additive-up-to-a-budget-limit. In these valuations, a set is valued at the sum of the values of its items, unless this sum is larger than a budget limit. In this last case it is valued at the budget limit.

DEFINITION 12. A valuation v is called additive with a budget limit if there exists a constant b, the budget limit, such that for all sets S of items, $v(S) = min(b, \sum_{i \in S} v(i))$.

PROPOSITION 2. *Every additive valuation with a budget limit is submodular.*

Proof. Assume $S \subseteq T$ and $x \notin T$. We have

$$v(x \mid T) = min(b, v(T) + v(\{x\})) - v(T) =$$

$$min(b - v(T), v(x)) \leq min(b - v(S), v(x)) = v(x \mid S).$$

∎

These type of valuations do not necessarily satisfy the gross substitutes property – see below Example 2.

4.1.3. Valuations based an underlying measure

The third class of valuations we want to present consists of valuations based on some underlying set with a measure on it. Let us first start with a concrete example. Consider a combinatorial auction for a set of spectrum licenses in overlapping geographical regions. Each geographical region contains a certain population, and a reasonable valuation for a set of licenses is the total population in the geographic area covered by the regions in the set. It turns out that such valuations are submodular. More generally, the overlapping regions may be arbitrary sets in some underlying base set, and the population count may be replaced by any measure on the base set.

DEFINITION 13. A valuation v is said to be based on the underlying measure μ on a base set Γ if there are sets $I_1 \ldots I_m \subseteq \Gamma$ such that for each set S of goods, $v(S) = \mu(\cup_{i \in S} I_i)$.

LEMMA 6. *Every valuation that is based on an underlying measure is submodular.*

Proof. Consider $S \subseteq T$, and an item x. Denote $\tilde{S} = \cup_{i \in S} I_i$, and similarly $\tilde{T}$. Clearly $\tilde{S} \subseteq \tilde{T}$. We now have $v(x \mid S) = \mu(I_x - \tilde{S}) \geq \mu(I_x - \tilde{T}) = v(x \mid T)$. ∎

4.2. Separation Examples

4.2.1. $OXS \neq GS$

EXAMPLE 1. Consider a set X of four items: $X = \{a, b, c, d\}$ and the valuation v defined by: the value of any singleton is 10 and the value of any other set is 19, except for the two sets $\{a, c\}$ and $\{b, d\}$ the value of which is 15.

CLAIM 2. *The valuation v is not in OXS.*

Proof. Suppose $v = v_1 \vee \ldots \vee v_n$. Since $v(\{a\}) = 10$, there is some i with $v_i(\{a\}) = 10$. Without loss of generality, assume $i = 1$. Since $v(\{b\}) = 10$, there is some i with $v_i(\{b\}) = 10$. Since $v(\{a, b\}) < 20$, for any $i \neq 1$, $v_i < 10$. We conclude that $v_1(\{b\}) = 10$. Similarly, $v_1(\{c\}) = v_1(\{d\}) = 10$. Consider now that $v(\{a, c\}) = 15$, therefore, since $v_1(\{a\}) = 10$, for any $i > 1$, $v_i(\{c\}) \leq 5$. Similarly, for any item x and any $i > 1$, $v_i(\{x\}) \leq 5$. But $v(\{a, b\}) = 19$ and therefore there must be $i \neq j$ such that $v_i(\{a\}) + v_j(\{b\}) = 19$, a contradiction. ∎

CLAIM 3. *The valuation v is in GS.*

Proof. (sketch) We will use the single improvement property (Lemma 2). Let $\vec{p}$ be any price vector and consider any non-preferred set A. Let $D \in D(v \mid \vec{p})$ be a preferred set. We need to show that we can improve the surplus of A by deleting at most one element and inserting at most one element. This requires a tedious case by case analysis that we omit, except for the only nontrivial case where $A = D^c$ and both are of size 2, e.g. $A = \{a, b\}$ and $D = \{c, d\}$. In this case either $\{a, d\}$ or $\{b, c\}$ will have a surplus greater than A's. ∎

4.2.2. GS ≠ SM

EXAMPLE 2. Consider the additive valuation with budget limit 4 on three elements: $v(1) = v(2) = 2$, $v(3) = 4$. Since the budget limit is 4, each set that contains at least two elements has a valuation of 4.

This valuation is submodular by Proposition 2, but it does not satisfy the gross substitutes property. Consider the price vector $p_1 = 0, p_2 = 1, p_3 = 2$. At these prices the preferred subset is $\{1, 2\}$ giving a surplus of 3. However, if we increase the price of item 1 to $p_1 = 2$, then the preferred subset is $\{3\}$ giving a surplus of 2, where any set that contains item 2 will have a surplus of at most 1. Thus we have reduced the demand for item 2.

4.2.3. SM ≠ XOS

EXAMPLE 3. Consider the symmetric valuation on three elements defined by $p_1 = 2$, $p_2 = 0$, $p_3 = 1$: a set of one or two elements is valued at 2, while the set of all three elements is valued at 3.

By definition, it is not downward sloping, therefore not submodular, by Proposition 1, but it is obtained by the following XOR-of-OR-of-singletons, where $(\{a\} : p)$ represents the singleton valuation that values any bundle containing item a at value p and any other bundle at value zero.

$$(\{1\} : 2) \oplus (\{2\} : 2) \oplus (\{3\} : 2) \oplus w$$

where $w = (\{1\} : 1) \vee (\{2\} : 1) \vee (\{3\} : 1)$.

4.2.4. $XOS \neq CF$

EXAMPLE 4. Assume a set X of three items and $v(S) = 2$ if $\mid S \mid = 1$, $v(S) = 3$ if $\mid S \mid = 2$ and $v(S) = 5$ if $\mid S \mid = 3$.

The valuation v is in CF since $2 + 2 \geq 3$ and $2 + 3 \geq 5$. We shall show that it cannot be expressed as a XOR-of-ORs-of-singletons. Assume that it is, and consider the OR clause that provides the valuation of 5 to the set of all three elements. This OR clause contains (at most) three singleton bids for the three items; if we take the two highest bids of these three we must get a valuation of at least $5 \cdot 2/3 > 3$, in contradiction to the valuation of any two elements being 3.

This is also an example for the fact that the class CF is not closed under marginal valuations. The marginal valuation v_W, where W contains any single item, gives $v_W(S) = 1$ if $\mid S \mid = 1$ and $v_W(S) = 3$ if $\mid S \mid = 2$ and is not in CF.

4.2.5. SM is not closed under OR

The following example will show that SM is not closed under OR.

EXAMPLE 5. Let $u(1) = 3$, $u(2) = 5$ and $u(3) = 3$, where u has a budget limit of 6. Let w be the additive valuation: $w(1) = 1$, $w(2) = 2$, $w(3) = 0$. Let $v = u \vee w$.

The valuation u is submodular by Proposition 2. One can see that: $v(\{1,2\}) = 6$ (u gets both), $v(\{2,3\}) = 6$ (u gets both). Therefore $v(\{1,2\}) + v(\{2,3\}) = 12$. But $v(\{1,2,3\}) = 8$ (u gets 1 and 3 and w gets 2), $v(2) = 5$ (u gets it), and $8 + 5 > 12$ and v is not submodular.

We know of two special cases in which the OR of submodular functions is submodular. The first concerns symmetric valuations.

PROPOSITION 3. *If v_1 and v_2 are symmetric submodular valuations then $v_1 \vee v_2$ is submodular (and symmetric).*

Proof. By proposition 1 and Nisan(2000), any symmetric submodular valuation is an OXS valuation. Since OXS is clearly closed under the OR operation we have that $v_1 \vee v_2$ is in OXS and thus also, by Theorem 6, is submodular. ∎

The second case concerns valuations with disjoint supports.

PROPOSITION 4. *If v_1 and v_2 are submodular valuations with disjoint supports (i.e., $v_i(x) > 0$ implies $v_j(x) = 0$) then $v_1 \vee v_2$ is submodular.*

Proof. For any set S: $(v_1 \vee v_2)(S) = v_1(S) + v_2(S)$. ∎

5. ALLOCATION

We now turn to the computational problem of allocating the items in a combinatorial auction in which all agents have submodular valuations. As a computational problem, we must first consider the format of the input, i.e. how are the valuation functions $v_1 \ldots v_n$ presented to the algorithm. In the most general case representing a valuation may require exponential size (in m, the number of items). We, on the other hand, are looking for algorithms that are polynomial in the relevant parameters, n and m. There are two possible approaches for obtaining efficient algorithms despite the exponential size of the input.

The first approach considers the general case, but where the valuations are presented to the algorithm as "valuation oracles" – black boxes that can be queried for the valuation of a set S, returning its valuation $v(S)$. In a mechanism, this corresponds to allowing bidders to send an arbitrary representation of the valuation function, as long as the valuations of sets can be efficiently computed from it[6]. The second approach fixes a representation, and provides allocation algorithms that require polynomial time in the size of the representation of the valuations in this format. Such algorithms will generally be as interesting as the strength of the representation format.

We present our main positive result in section 5.3, in the general terms of valuation oracles. Our main negative result, the NP-hardness of allocation for submodular valuations, presented in subsection 5.2, holds for a special case (additive valuations with a budget limit) that has a simple short representation. Both our results are therefore presented in the strongest possible manner. Concrete lower bounds for the valuation oracle representation are derived in Nisan and Segal(2001). We first review the case of GS agents, which is polynomial time solvable.

5.1. The Case of Gross Substitutes

The case of a combinatorial auction where all valuations satisfy the gross substitutes property can be solved in polynomial time. Since Walrasian equilibria exist in this case by Kelso and Crawford(1982) (see definition 8 and theorem 2 above), they can be found and an optimal allocation results. Let us look more closely at the computational process of finding this Walrasian equilibrium. The following procedure of Kelso and Crawford(1982) yields prices and an allocation that is arbitrarily close to optimal:

1. Initialize all item prices to 0.
2. Repeat the following procedure: compute a preferred set for each bidder at these prices and increase by a small amount ϵ the price of all items that are demanded by more than one agent.

[6]Surprisingly, achieving this is not totally trivial – see section 3.6 of Nisan(2000).

3. Stop when each item is demanded by at most one agent, and let the allocation be the preferred sets at these prices.

This procedure produces an allocation whose distance from optimal depends only on ϵ and whose running time is polynomial in n, m, ϵ^{-1}. This algorithm is naturally implemented as a mechanism, in some cases is actually incentive compatible, and in the general case can serve as a basis for an incentive compatible mechanism as in Ausubel(2000). But this procedure only provides an approximation (it is actually a "FPAS" – fully polynomial approximation scheme) and not an exact solution.

An exact solution can be computed in polynomial time using linear programming. Indeed, the optimization problem can be described as an integer-programming problem whose LP relaxation has an integral solution (see Bikhchandani and al.(2001)). This linear program has an exponential number of variables, but only a polynomial number of constraints. One can therefore solve the problem using a separation-based algorithm on the dual (see Nisan and Segal(2001) for more details). This LP algorithm requires only a polynomial number of calls to a *demand oracle* that, given a GS valuation v and a price vector $\vec{p}$, computes a preferred set $S \in D(v \mid \vec{p})$ at these prices.

The fact that allocation among GS valuations can be computed in polynomial time follows also from the algorithms for calculating the convolution of $M^{\#}$-concave functions developed in Murota(1996), Murota(2000) and Murota and Tamura(2001).

THEOREM 9. *An optimal allocation among GS valuations can be found in polynomial time, if the valuations themselves can be computed in polynomial time.*

A special case of interest is when all valuations in the auction are OXS valuations. If these valuations are given using the OR of XOR of singletons representation, then the allocation problem reduces to a matching problem in a bipartite graph and can thus be computed in polynomial time by Tarjan(1983), Fredman and Tarjan(1987).

5.2. NP-hardness of an exact solution among submodular players

The problem of finding an optimal allocation in combinatorial auctions is known from Rothkopf and al.(1998) to be NP-hard even if bidders are single-minded, i.e., place only one bid each. Is the problem any easier if the bidders are assumed to have submodular valuations? The answer is negative. Finding an optimal allocation is still an NP-hard problem, even if all valuations are additive with budget limit. Note that, in such a case, all players valuations may be succinctly expressed.

THEOREM 10. *Finding an optimal allocation in a combinatorial auction with two valuations that are additive with a budget limit is NP-hard.*

Proof. We will reduce from the well-known NP-complete problem "Knapsack" (see Garey and Johnson(1979)): Given a sequence of integers $a_1 \ldots a_m$, and a desired total t, determine whether there exists some subset S of the integers whose sum is t, $\sum_{i \in S} a_i = t$. Given an input of this form, construct the following two valuations on m items:

- The first valuation is additive giving the price a_i to each item i: $v_1(S) = \sum_{i \in S} a_i$.
- The second valuation is additive with a budget limit of $2t$, and gives the price $2a_i$ to each item i: $v_2(S) = 2 \cdot \min(t, \sum_{i \in S} a_i)$.

Fix an allocation of S to valuation 2 and S^c to valuation 1 and consider the 3 cases: $\sum_{i \in S} a_i < t$, $\sum_{i \in S} a_i = t$, and $\sum_{i \in S} a_i > t$. Denote $f = \sum_i a_i$. If $\sum_{i \in S} a_i = t$ then $t + \sum_{i \in S^c} a_i = f$ and

$$v_1(S^c) + v_2(S) = \sum_{i \in S^c} a_i + 2t = f + t.$$

If $\sum_{i \in S} a_i < t$ then

$$v_1(S^c) + v_2(S) = \sum_{i \in S^c} a_i + 2 \sum_{i \in S} a_i = f + \sum_{i \in S} a_i < f + t.$$

If $\sum_{i \in S} a_i > t$ then $\sum_{i \in S^c} a_i + t < f$, and

$$v_1(S^c) + v_2(S) = \sum_{i \in S^c} a_i + 2t < f + t.$$

Thus we see that the auction has an allocation $(S^c : S)$ with value $f + t$ if the knapsack problem has a positive answer S, and otherwise the allocation has a lower value. ∎

Two comments should be made here: first, this NP-hardness result is for specific, succinctly defined, submodular valuations. The case of general submodular valuations given as an oracle is harder in a stricter sense, that of requiring exponentially many oracle queries as shown in Nisan and Segal(2001). Second, this result is for allocation among several valuations. A related well studied question is the maximization of a single submodular function (not necessarily monotone). This problem is well known to be NP-hard for some specific succinctly defined functions, and to require exponentially many queries in the general oracle model, see Jensen and Korte(1982), Lovász(1980), and Lovász(1983).

5.3. A 2-approximation

Our main algorithmic result is a 2-approximation algorithm for combinatorial auctions in which all valuations are submodular. The result does not rely on any specific representation of submodular functions: it only assumes that one can effectively compute valuations of subsets.
Input: $v_1, \ldots, v_n$ - submodular valuations, given as black boxes.
Output: An allocation (partition of the items) $S_1, \ldots, S_n$ which is a 2-approximation to the optimal allocation
Algorithm

1. Set $S_1 = S_2 = \ldots = S_n \leftarrow \emptyset$.
2. For $x = 1 \ldots m$ do:
   (a) Let j be the bidder with highest value of $v_j(x \mid S_j)$.
   (b) Allocate x to bidder j, i.e. $S_j \leftarrow S_j \cup \{x\}$.

The algorithm obviously requires only a polynomial number of operations and calls to valuation oracles for v_j since $v_j(x \mid S_j) = v_j(S_j \cup \{x\}) - v_j(S_j)$.

THEOREM 11. *If all v_i are submodular valuations then the greedy algorithm above provides a 2-approximation to the optimal one.*

Proof. Let Q be the original problem and define Q' to be the problem on the $m-1$ remaining items after item 1 is removed: i.e., item 1 is unavailable and v_j is replaced by v'_j with $v'_j(S) = v(S \mid \{1\}) = v(S \cup \{1\}) - v(\{1\})$, where j is the player to which item 1 was allocated. All other valuations v_i, $i \neq j$ are unchanged. Notice that the algorithm above may be viewed as first allocating item 1 to j and then allocating the other elements using a recursive call on Q'.

Let us denote by $ALG(Q)$ the value of the allocation produced by this algorithm, and by $OPT(Q)$ the value of the optimal allocation. Let $p = v_j(\{1\})$. By the definition of Q', it is clear that $ALG(Q) = ALG(Q') + p$. We will now show that $OPT(Q) \leq OPT(Q') + 2p$. Let $S = (S_1, \ldots, S_n)$ be the optimal allocation for Q, and assume that $1 \in S_k$, i.e., item 1 is allocated to bidder k by the algorithm above. Let $S' = (S'_1, \ldots, S'_n) = (S_1, \ldots, S_k - \{1\}, \ldots, S_n)$ be the solution to Q' obtained from S by ignoring item 1. Let us compare the value of S' in Q'(which is a lower bound on $OPT(Q')$) to the value of S in Q (which is exactly $OPT(Q)$). All players except k get the same allocation and all players except j have the same valuation. Without loss of generality, assume $k \neq j$. Player k loses at most $v_k(\{1\})$ since v_k is submodular. But $v_k(\{1\}) \leq v_j(\{1\}) = p$ and player k looses at most p. Player j looses at most p since, by monotonicity of v_j, $v'_j(S_j) = v_j(S_j \cup \{1\}) - v_j(\{1\}) \geq v_j(S_j) - p$. Therefore

$OPT(Q') \geq OPT(Q) - 2\,p$. The proof is concluded by induction on Q' since, by lemma 1, Q' also consists of submodular valuations:

$$OPT(Q) \leq OPT(Q') + 2\,p \leq$$

$$2\,ALG(Q') + 2\,p = 2\,ALG(Q).$$

∎

Looking at the proof one may see that many variants of the above algorithm also produce a 2-approximation. In particular the items x may be enumerated in any order in the outer loop. One may think of several heuristics for choosing the next x. For example: take the item x that maximizes the difference between the first and second values of $v_j(x \mid S_j)$.

An alternative proof for a variant of this algorithm may be obtained using the results of Fisher and al.(1978) regarding the maximization of a single submodular valuation over a matroid. Indeed the winner determination problem in a combinatorial auction is a maximization problem over a matroid, and this does not seem to have been observed previously. Consider the set $P = N \times X$, where N is the set of agents and X is the set of items. A subset $S \subseteq P$ is a partial allocation iff each item is included in at most one element of S. The family of partial allocations is a matroid. The function to be maximized is the sum of $n =\mid N \mid$ valuations. In our case, each valuation is submodular and therefore their sum is also submodular, and the existence of a polynomial-time 2-approximation algorithm follows from Theorem 2.1 of Fisher and al.(1978). The NP-hardness result of section 5.2 shows we are dealing with a hard case of submodular monotonic function maximization over a matroid.

The family of greedy algorithms we consider is wider than the one considered by those authors since they request a specific ordering of the items and we do not, leaving more space for heuristic considerations. In section 7 we shall generalize this result to valuations that are not submodular, and therefore fall outside the scope of Fisher and al.(1978).

Recent work (Lehmann(2002)) shows that the allocation produced by this algorithm is not only a 2-approximation to the optimal integer allocation but also a 2-approximation to the optimal *fractional* allocation.

It is easy to see that our bound is sharp and the algorithm above may provide only a 2-approximation even for submodular valuations: take $v_1(\{1\}) = v_1(\{2\}) = v_1(\{1,2\}) = v_2(\{1\}) = v_2(\{1,2\}) = 1$ and $v_2(\{2\}) = 0$.

Open Problem: Does any polynomial algorithm provide a better approximation ratio?

Conforti and Cornuéjols(1984) sharpened the results of Fisher and al.(1978). Their results offer better bounds when the submodular valuations are known to be close to linear, but not in the general case. They also provide better bounds when the matroid is close to uniform, but our matroid has

very small girth (1) and large rank (m). We have also shown above that our bound is sharp. The results of Nisan and Segal(2001) provide a weak lower bound showing that achieving an approximation ratio of better than $1+1/m$ requires an exponential number of queries to the valuation oracles.

5.4. Strategic considerations

The approximation algorithm presented above may be viewed as a simple sequence of auctions of the single items, one by one. Simplistic bidders whose strategies do not take the future into account should indeed value an item i at its current marginal value for the bidder. Thus if each auction in this sequence is designed to be incentive compatible (e.g. a second price auction) and if all bidders follow this simplistic bidding strategy (myopic bidders) then the desired 2-approximate allocation would be obtained. In this subsection we ask whether there is a payment scheme to be used with this allocation algorithm that guarantees incentive-compatibility of the complete auction, i.e. that will reach this allocation when the bidders are rational and not myopic. The answer is negative.

An example will show that no payment scheme can make the greedy allocation scheme for sub-modular combinatorial auctions a truthful mechanism. In this example, the greedy scheme allocates the most expensive items first. We do not know whether the result may be generalized to any greedy scheme. The spirit of the proof is similar to that of Section 12 of Lehmann and al.(forthcoming).

EXAMPLE 6. Two goods: a and b. Two sub-modular players: Red and Green. Red declares 10 for a, 6 for b and 11 for the set $\{a, b\}$. Notice this is a sub-modular declaration, but this is not crucial. Green has two personalities:

- Green1 declares 11 for a, 10 for b and 18 for the set $\{a, b\}$.
- Green2 declares 9 for a, 10 for b and 19 for the set $\{a, b\}$.

Notice both are sub-modular declarations.

The allocation between Red and Green1 goes in the following way: a is allocated first to Green1 (Green's 11 vs. Red's 10) and then b is allocated to Green1 (Green's 18-11 vs. Red's 6). Notice that if b had been allocated first Green would have obtained b but not a. Green1 is therefore allocated the set $\{a, b\}$ and pays a sum p. Notice that, since all declarations are fixed, p is a number; it does not depend on anything. The allocation between Red and Green2 goes a different way: any ordering gives a to Red and b to Green. Green2 pays q (just a number).

If the mechanism is truthful and Green is Green1, it must be the case that Green cannot gain by disguising himself as Green2: $18 - p \geq 10 - q$. If the mechanism is truthful and Green is Green2, it must be the case that

Green cannot gain by disguising himself as Green1: $10 - q \geq 19 - p$. A contradiction.

6. FALSE-NAME BIDS

This section deals with the following strategic manipulation problem that is inherent in combinatorial auctions that use the Vickrey-Clark-Groves payment rules (Vickrey(1961),Clarke(1971),Groves(1973) or see MacKie-Mason and Varian(1994)). The VCG rules dictate that a player that is allocated a set S of items pays the external cost to society of his allocation, i.e., his payment is $u(X) - u(X - S)$, where X is the set of all items, and u is the combined (by OR) valuations of all other bidders (i.e. the valuation obtained by the value of the optimal allocation of items among the other bidders). A disturbing observation made in Sakurai and al.(1999) is that in many cases a bidder can manipulate a VCG combinatorial auction and reduce his payment by splitting his bid and placing two separate bids under *false-names*. For example, if two items A and B are offered and my valuation for the pair $\{A, B\}$ is 6, while another bidder values the pair at 5, the VCG rules will set my payment to 5. If, instead, I place two separate bids, $\{A\}$ for 4 and $\{B\}$ for 4, then one may easily calculate that each of these *false-name bidders* pays $1 = 5 - 4$ for a total payment of 2.

This issue was further discussed in Yokoo and al.(2000a, 2000b, 20001). They showed that such false-name bidding cannot be profitable for any agent if the surplus function is concave, i.e., submodular over agents. This condition is a strengthening of the *agents are substitutes* condition first considered in Shapley(1962) and then in Bikhchandani and Ostroy(forthcoming). We show that a different condition also guarantees that false-name bidding is not profitable. If the aggregate valuation of all other agents is submodular *in the items*, one cannot benefit from placing false-name bids. We then discuss this condition and show that the result is sharp: false-name attacks may be profitable even if all agents are submodular.

THEOREM 12. *A player cannot benefit from placing false name bids in a combinatorial auction using the VCG rules, whenever the combined valuation of all other players is submodular.*

Proof. We shall show that two players, one of them bidding a valuation v_1 and the other a valuation v_2 will pay *no less* than a single player bidding $v_1 \vee v_2$, as long as the combined valuation function of the other players is submodular.

Consider the set of bids $v_1, v_2, u_1, \ldots, u_m$ and let S_1 be the set allocated to player 1, S_2 be the set allocated to player 2, and T be the set allocated to all the u_i bidders together: thus the total set of goods is $T \cup S_1 \cup S_2$. Let $u = u_1 \vee \ldots \vee u_m$.

The VCG rules (GVA auction) specify that player 1 will pay:

$$(u \vee v_2)(T \cup S_1 \cup S_2) - (u \vee v_2)(T \cup S_2).$$

Since the optimal allocation of $T \cup S_2$ among the valuations u and v_2 allocates T to u and S_2 to v_2, we have that $(u \vee v_2)(T \cup S_2) = u(T) + v_2(S_2)$. By considering the allocation of S_2 to v_2 and $T \cup S_1$ to u, we can bound

$$(u \vee v_2)(T \cup S_1 \cup S_2) \geq u(T \cup S_1) + v_2(S_2).$$

We thus get that player 1 pays at least $u(T \cup S_1) - u(T)$. Similarly player 2 will pay at least $u(T \cup S_2) - u(T)$.

Consider, on the other hand what happens when instead of v_1 and v_2 a single $v_1 \vee v_2$ bid is submitted. The allocation to this bidder is exactly $S_1 + S_2$. The VCG payment of this player will be $u(T \cup S_1 \cup S_2) - u(T)$. The submodularity of u directly implies that this is less than or equal to the sum of the payments of players 1 and 2 in the previous case. ∎

As a simple corollary we obtain.

COROLLARY 3. *A player cannot benefit from placing false name bids in a combinatorial auction using the VCG rules, where all valuations satisfy the gross substitutes property.*

Proof. According to theorem 5, the combined valuation (OR) of GS valuations is GS and thus is also submodular. ∎

A slightly weaker result could have been proven by showing that the aggregate utility (OR combination) of GS valuations is concave in the agents and using Proposition 3 of Yokoo and al.(2000a). But Common Knowledge of the fact that all players are GS would have been required.

Corollary 3 also implies a previous result of Sakurai and al.(1999) showing that in a Vickrey multi-unit auction of identical items in which all players have a downward sloping valuations, no player can benefit from placing false-name bids.

Could Theorem 12 assume, instead, that all players have a submodular valuation? The following example shows that this is not the case. The following example builds on Example 5 and presents an auction in which each player is submodular, but the combined valuation of the opponents is not and one can benefit from false-name bids.

EXAMPLE 7. Red has budget limit 6 and values: $red(a) = red(c) = 3$, $red(b) = 5$. Blue has unbounded budget: $blue(a) = 1, blue(b) = 2, blue(c) = 0$. Let v be the combined valuation, $v = red \vee blue$. We have that $v(abc) = 8$, $v(ab) = v(ac) = v(bc) = 6$. Green has unbounded budget: $green(a) = 2$, $green(b) = 0$, $green(c) = 5$.

If Green acts as himself, he gets $\{a, c\}$ (utility 7) and Red gets $\{b\}$ (utility 5) for a total of 12, unbeatable. Green pays: 8-5 = 3.

But, assume that Green acts under two different identities $G1(a) = 2$ and $G2(c) = 5$. Obviously $G1$ gets $\{a\}$ and $G2$ gets $\{c\}$. $G1$ pays: 11-10=1.

$G2$ pays: 8-7=1. On the whole $G1$ and $G2$ together pay only 2 which is strictly less than 3.

This example also shows that the "agents are substitutes" condition of Shapley(1962) and Bikhchandani and Ostroy(forthcoming) does *not* follow from the fact that all agents have submodular valuations. Let T be the set of agents containing: Red, Blue, G1 and G2. We have seen that $v(T) = 12$. Let S be the set containing G1 and G2. We have seen that $v(T - S) = 8$. But $v(T - \{G1\}) = 11$ and $v(T - \{G2\}) = 8$. We see that $v(T) - v(T - S) < (v(T) - v(T - \{G1\})) + (v(T) - v(T - \{G2\}))$.

7. BOUNDED COMPLEMENTARITY

While the definition of submodular valuations is theoretically clean, in many *real world* situations, valuations cannot be assumed to be submodular. In many cases, valuations exhibit complementarities. Complementarities are, indeed, one of the main reasons for the development of parallel or combinatorial auctions. In much of traditional economic theory, decreasing marginal values, or convexity, have been assumed more for mathematical expediency reasons than because they fit the data. The submodularity assumption will now be weakened in a way that allows it to encompass complementarities and nevertheless retains its mathematical structure.

In many cases it can be argued that marginal valuations should be *rather* decreasing, i.e., should not be increasing *too much*, or that if complementarities exist, they are not too large and may be bounded in some way. In this section we formalize this idea and then observe that in such cases the greedy allocation algorithm still produces provably good results. Thus our results assuming submodularity have wider implications.

The definitions and results of Section 2.4 can be generalized to deal with valuations that exhibit a limited amount of complementarity. We assume $a \geq 1$ is a real number. The number a is a measure of how close we are to complement-freeness or to submodularity. The case $a = 1$ is the one considered above in this paper: complement-freeness and submodularity. The larger the number a the looser our assumptions.

DEFINITION 14. A valuation v is said to exhibit a-bounded complementarities if, for any set A and item x:

$$v(A \cup \{x\}) \leq v(A) + a\, v(\{x\}).$$

Complementarities are bounded multiplicatively by the number a.

DEFINITION 15. A valuation v is said to be a-submodular if and only if for every subset W of items, the marginal valuation $v(\cdot \mid W)$ exhibits a-bounded complementarities.

The proofs of the following results are similar to those of Section 2.4 and will be omitted.

PROPOSITION 5. *A valuation v is a-submodular if and only if any one of the following equivalent propositions holds.*

- *For any $x \in X$ and $S, T \subseteq X$, such that $S \subseteq T$ and $x \notin T$: $a\, v(x \mid S) \geq v(x \mid T)$.*
- *For any $S, T, V \subseteq X$, such that $S \subseteq T$: $a\, v(V \mid S) \geq v(V \mid T)$.*
- *For any $A, B \subseteq X$: $v(A) + a\, v(B) \geq v(A \cup B) + a\, v(A \cap B)$.*

The 2-approximation result of Section 5 generalizes to a $1+a$-approximation.

THEOREM 13. *The greedy algorithm provides a $(1 + a)$ approximation to the optimal one if all valuations are a-submodular.*

Notice that the problem we solved is of maximizing a function that is *not* submodular over a matroid. We do not know whether this may have useful implications for other such problems.

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