Nonmonotonic reasoning: from finitary relations to infinitary inference operations *

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Abstract

A. Tarski [22] proposed the study of infinitary consequence operations as the central topic of mathematical logic. He considered monotonicity to be a property of all such operations. In this paper, we weaken the monotonicity requirement and consider more general operations, inference operations. These operations describe the nonmonotonic logics both humans and machines seem to be using when infering defeasible information from incomplete knowledge. We single out a number of interesting families of inference operations. This study of infinitary inference operations is inspired by the results of [12] on nonmonotonic inference relations, and relies on some of the definitions found there.

1 Introduction

Since Tarski [22] and Gentzen [9], logicians have had to choose between two possible frameworks for the study of logics. A set $\mathcal{L}$ of formulas being given, one may model a logic by a mapping $\mathcal{C} : 2^{\mathcal{L}} \longrightarrow 2^{\mathcal{L}}$ and for every subset X of $\mathcal{L}$, $\mathcal{C}(X)$ is understood to be the set of all the consequences of the set X of assumptions. But one may think that only finite sets of assumptions have well-defined consequences, in which case one generally represents a logic by a

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relation $\vdash$ between a finite set of formulas on the left and a single formula on the right. For a finite set A of formulas and a formula b, the relation $A \vdash b$ holds when b is a consequence of the finite set A of assumptions. It turns out that, in the study of mathematical logic, the mappings $\mathcal{C}$ of interest are monotonic, i.e., satisfy $\mathcal{C}(X) \subseteq \mathcal{C}(Y)$ whenever $X \subseteq Y$. Very often they are also compact, i.e., whenever $a \in \mathcal{C}(X)$, there exists a finite subset A of X such that $a \in \mathcal{C}(A)$. For compact monotonic mappings, the two frameworks are equivalent since, given an infinitary consequence operation $\mathcal{C}$ one may define a finitary relation $\vdash$ by:

$$A \vdash a \Leftrightarrow a \in \mathcal{C}(A)$$

and, given a (finitary) relation $\vdash$, one may define an operation $\mathcal{C}$ by:

$$\mathcal{C}(X) = \{a \mid \text{there exists a finite set} A \subseteq X \text{ such that } A \vdash a\}$$

and this provides a bijection between the two formalisms. The choice of a framework for a study of nonmonotonic logics has more serious consequences, since, even for compact mappings, there is no bijection as in the monotonic case. At this point, we do not know whether there is a reasonable notion of *pseudo-compactness* for which any finitary operation has a unique *pseudo-compact* extension. It is only recently that the general study of nonmonotonic logics has begun and both frameworks have been used. So far, [8], [12],[13], [15], and [16] opted for the finitary framework, whereas [17] and [18] opted for the infinitary framework. The present work will build a bridge between the two frameworks. A different comparison of those two approaches may be found in [1]. More precisely, since the finitary framework has, so far, been richer in technical results and provided more insight in defining interesting families of relations, this paper will extend to the infinitary framework most of the notions and results obtained in the finitary one. Mainly, three types of question will be studied. First, we shall try to define families of infinitary inference operations that are similar to the finitary families defined in [12]. In doing so, we shall notice some unsuspected complications stemming from the fact that finitary properties that are equivalent sometimes have infinitary analogues that are not. Then, we shall deal with the problem of extending relations to infinitary operations. This is a crucial question in deciding which framework to prefer. If there are relations that cannot be extended, then the study of infinitary operations will be of little use for learning about relations. It will be shown that, for each one of the families of interest, every relation has a suitable extension, but, in general, more than one such extension. In fact, we shall even propose a canonical way of extending operations that maps (almost) each finitary family into its analogue. The third type of questions we shall deal with is the obtention of representation theorems that sharpen the representation theorems of [12] and [16].

Another aspect of this work must be clarified here. One may consider the underlying language of formulas $\mathcal{L}$ in a number of possible ways. One may suppose nothing about it, as did D. Makinson in his first efforts, and, more recently, D. Lehmann in [14]. But many interesting families of operations may be defined only if one assumes that $\mathcal{L}$ comes with a monotonic consequence operation and is closed under certain connectives. Although most of the results presented here may be shown to hold even in the case no connectives are known to be in the language, or under minimal assumptions concerning the existence of connectives, (see [5] for such results), for brevity's sake, we shall suppose that our language $\mathcal{L}$ is a classical propositional one, equipped with the usual connectives, as was done in [12].

2 Plan of this paper

This paper describes and studies a number of families of inference operations, roughly from the most general to the most restricted. First, in Section 3, we present the background we need concerning monotonic consequence operations, and inference operations. Then, four principal families of inference operations are presented.

In Section 4, we consider the *cumulative* operations. Cumulative operations are what D. Makinson called supraclassical cumulative inference operations in [17]. The finitary version of this definition characterizes the cumulative relations studied in [12]. The main results of this section deal with the problem of extending (finitary) cumulative inference relations into infinitary cumulative operations. Two such extensions are described. The first one, due to D. Makinson, is the smallest possible cumulative extension of a given cumulative relation. The second one, termed *canonical* extension extends to cumulative relations the construction first presented in [6] for preferential relations. We conclude this section by model-theoretic considerations: as in the finitary case, any cumulative operation may be defined by a suitable model. This representation theorem is a relatively straightforward sharpening of the corresponding result of [12].

Section 5 defines *distributive* operations. Distributivity is the property that corresponds, in the infinitary framework, to the Or rule of [12]. A weak form of distributivity is shown to be equivalent to distributivity. Properties of distributivity are studied. The lack of a representation theorem for distributive operations seems to indicate this family is not as well-behaved as others.

In Section 6, a more restricted family of cumulative operations, the *deductive* operations, is defined. They are the operations that satisfy an infinitary version of the S rule of [12]. We show that, although the rules Or and S are equivalent for cumulative relations, their infinitary counterparts are not: all deductive operations are distributive but there are distributive operations that are not deductive. We show that the canonical extension of a preferential relation is deductive. We characterize monotonic deductive operations as translations of the operation of logical consequence. We, then, study Poole systems, quote some results of D. Makinson and show that any finite Poole system without constraints defines a deductive operation that is the canonical extension of its finitary restriction. A representation theorem for deductive operations is established, that is a non-obvious generalization of the corresponding result of [12].

In Section 7, we deal with an important family of deductive operations, those that satisfy the infinitary version of Rational Monotonicity. We show that the canonical extension of a rational relation provides a rational operation and conclude with a representation theorem for rational operations in terms of modular models. Section 8 is a conclusion.

3 Background

Let $\mathcal{L}$ be a propositional language: i.e., $\mathcal{L}$ is the set of all propositional formulas on a given non-empty set of propositional variables. For any set X of propositions, $\mathcal{C}n(X)$ will be the set of all logical consequences of X. We shall use the symbol $\subseteq_f$ to mean *is a finite subset of*. As is usually done, for $X, Y \subseteq \mathcal{L}$ and $a \in \mathcal{L}$, we shall write $\mathcal{C}n(X, Y)$ instead of $\mathcal{C}n(X \cup Y)$, $\mathcal{C}n(X, a)$ instead of $\mathcal{C}n(X \cup \{a\})$ and $\mathcal{C}n(a)$ instead of $\mathcal{C}n(\{a\})$.

Definition 1 *A set of formulas T that is closed under $\mathcal{C}n$, i.e., such that $T = \mathcal{C}n(T)$ is called a theory.*

We shall use the fact that the intersection of a family of theories is a theory. For $X, Y \subseteq \mathcal{L}$ we shall say that X is *consistent* iff $\mathcal{C}n(X) \neq \mathcal{L}$ and that X is *consistent with* Y iff $\mathcal{C}n(X, Y) \neq \mathcal{L}$. A set is *inconsistent* iff it is not consistent. For $X, Y \subseteq \mathcal{L}$, we shall use the notation $X \vee Y$ to denote the set $\{x \vee y \mid x \in X, y \in Y\}$. Note that $\mathcal{C}n(X \vee Y) = \mathcal{C}n(X) \cap \mathcal{C}n(Y)$.

We shall now introduce our definitions and notations for inference operations. Let S be a set, 2^S denotes the set of all subsets of S. We shall consider infinitary operations (in short operations) $\mathcal{C} : 2^{\mathcal{L}} \longrightarrow 2^{\mathcal{L}}$.

Definition 2

1. *An operation $\mathcal{C}$ is* left absorbing *iff, for any $X \subseteq \mathcal{L}$, $\mathcal{C}(X)$ is a theory.*
2. *An operation $\mathcal{C}$ is* right absorbing *iff, for any $X, Y \subseteq \mathcal{L}$, if $\mathcal{C}n(X) = \mathcal{C}n(Y)$, then $\mathcal{C}(X) = \mathcal{C}(Y)$. Thus, $\mathcal{C}$ is right absorbing iff, for any $X \subseteq \mathcal{L}$, $\mathcal{C}(\mathcal{C}n(X)) = \mathcal{C}(X)$.*

Definition 3 *An operation that is left-absorbing, right-absorbing and satisfies Inclusion (i.e., $X \subseteq \mathcal{C}(X)$) is called an* inference operation.

Notice that any left-absorbing operation $\mathcal{C}$ that satisfies Inclusion is supraclassical, i.e., for any set of formulas X, $\mathcal{C}n(X) \subseteq \mathcal{C}(X)$. We shall use, for inference operations, the same notations we use for consequence operations and write, for example $\mathcal{C}(X, Y)$ instead of $\mathcal{C}(X \cup Y)$. Inference operations represent ways of infering defeasible conclusions from incomplete information.

Left absorption means that the logical consequences of the *adventurous* inferences made by $\mathcal{C}$ are already themselves obtainable by $\mathcal{C}$. An inference operation is right absorbing if its inferences do not depend on the form of the assumptions but only on their logical meaning. As remarked in [18], absorption properties seem to be necessary characteristics of *logical* as opposed to *procedural* approaches.

Right absorption means that the image of any set of formulas is a theory. Left absorption means that the image of any set X is the same as that of its associated theory $\mathcal{C}n(X)$. We may, therefore, consider an inference operation as an arbitrary mapping of theories into theories that satisfies Inclusion, i.e., for any theory T, $T \subseteq \mathcal{C}(T)$. We shall do so freely. Another definition will be handy.

Definition 4 *Let $\mathcal{C}$ be an inference operation and X a set of formulas. We shall say that X is $\mathcal{C}$-consistent iff $\mathcal{C}(X) \neq \mathcal{L}$.*

4 Cumulative operations

4.1 Definition and first properties of cumulative operations

We shall define now the family of cumulative operations and show that it is the exact analogue of the family of cumulative relations of [12]. Historically, the study of cumulative operations was launched independently in two separate efforts. D. Makinson defined and studied cumulative infinitary operations, without assuming the existence of connectives. S. Kraus and D. Lehmann defined and studied cumulative finitary relations, in the framework of classical propositional calculus. This paper studies infinitary operations in the framework of classical propositional calculus. The terminology is mainly Makinson's.

Definition 5 *An operation $\mathcal{C}$ is* cumulative *iff it is an inference operation and satisfies the following two properties. For any $X, Y \subseteq \mathcal{L}$ such that $Y \subseteq \mathcal{C}(X)$:*

(Cut)	$\mathcal{C}(X, Y) \subseteq \mathcal{C}(X)$
(Cautious Monotonicity)	$\mathcal{C}(X) \subseteq \mathcal{C}(X, Y)$.

The definition of infinitary cumulative inference operations could have been given in one go, by saying that, under the assumptions, $\mathcal{C}(X, Y) = \mathcal{C}(X)$, but we prefered to define the two properties of Cut and Cautious Monotonicity. The following summarizes some easy results most of which appear in [17].

Theorem 1 *Any operation that satisfies Supraclassicality, Cut and Cautious Monotonicity is an inference operation, and therefore cumulative. An inference operation $\mathcal{C}$ is cumulative iff for all subsets $X, Y \subseteq \mathcal{L}$ such that $Y \subseteq \mathcal{C}(X)$ and $X \subseteq \mathcal{C}(Y)$ one has $\mathcal{C}(X) = \mathcal{C}(Y)$.*

We shall not try here to justify our interest in cumulative operations. The reader may wish to consult [12] and [18] for motivation. In the first reference, he will also find the definition of the rules used for defining families of nonmonotonic inference relations: Reflexivity, Left Equivalence, Right Weakening, Cut, Cautious Monotonicity, Loop, Or and S. Those definitions will not be repeated here. It is not clear, though, whether relations or infinitary operations are the topic of choice. We shall, therefore, devote, now, our efforts to studying the relation existing between those. It is clear that, given any infinitary cumulative operation $\mathcal{C}$, its restriction to singletons provides a cumulative relation, in the following way:

$$a \mathrel{|\!\sim} b \text{ iff } b \in \mathcal{C}(\{a\}). \tag{1}$$

A natural question is: may all cumulative relations be obtained in this manner? An infinitary operation $\mathcal{C}$ is called an *extension* of $\mathrel{|\!\sim}$ if (1) holds. Our natural question may now be rephrased as: do all cumulative relations have a cumulative extension? This is an important question because a positive answer will enable us to prove certain results for infinitary operations and then apply them to relations. As an example, Theorem 5 will provide us with a strengthening of the completeness part of Theorem 3.25 of [12]. The next sections will provide a positive answer to the question. In fact, we propose two different ways of extending cumulative relations, showing that a cumulative relation may have many different cumulative extensions.

4.2 The smallest cumulative extension of a cumulative relation

The first construction we shall present is due to D. Makinson. Since the intersection of a family of cumulative operations is clearly cumulative, if a relation has some extension, it has a smallest extension. We shall describe this extension.

Let $\mathrel{|\!\sim}$ be a cumulative relation. We are looking for a cumulative operation $\mathcal{C}$ such that $\mathcal{C}(x) = \mathcal{F}(x) \stackrel{\text{def}}{=} \{y \mid x \mathrel{|\!\sim} y\}$ for any proposition x. Let $X \subseteq \mathcal{L}$. Suppose that there is a proposition a such that $a \in \mathcal{C}n(X) \subseteq \mathcal{F}(a)$ (in such a case we shall say that X is of the *first type*). Then, for any suitable $\mathcal{C}$, one must have $\mathcal{C}n(X) \subseteq \mathcal{C}(a)$ and, by cumulativity, $\mathcal{C}(a, \mathcal{C}n(X)) = \mathcal{C}(a)$. But $a \in \mathcal{C}n(X)$ and $\mathcal{C}(\mathcal{C}n(X)) = \mathcal{C}(X)$ by right absorption. We conclude that $\mathcal{C}(X) = \mathcal{C}(a) = \mathcal{F}(a)$. There is therefore only one possibility in defining $\mathcal{C}(X)$ for an X of the first type.

It turns out that, for such an X, one may define unambiguously $\mathcal{C}(X)$ to be $\mathcal{F}(a)$ for any $a \in \mathcal{C}n(X)$ such that $X \subseteq \mathcal{F}(a)$. Indeed, for any two propositions $a_i, i = 0, 1$, such that $a_i \in \mathcal{C}n(X) \subseteq \mathcal{F}(a_i)$, one has $a_0 \in \mathcal{F}(a_1)$ and $a_1 \in \mathcal{F}(a_0)$. Since $\mathrel{|\!\sim}$ is cumulative we have $\mathcal{C}(a_0) = \mathcal{C}(a_1)$. There is therefore exactly one way to define the extension of $\mathrel{|\!\sim}$ on sets of the first type. We are left with a choice about the definition of $\mathcal{C}(X)$ for X's that are not of the first type. The *smallest* possible choice is to set: $\mathcal{C}(X) = \mathcal{C}n(X)$ in such cases. We shall show that the $\mathcal{C}$ obtained in this manner is cumulative and extends $\mid\!\sim$. It is clear that it is the smallest possible such operation.

Theorem 2 *For any cumulative relation $\mid\!\sim$, the operation $\mathcal{C}$ defined by:*

$$\mathcal{C}(X) = \begin{cases} \mathcal{F}(a) & \textit{if there is a proposition } a \in \mathcal{C}n(X) \textit{ such that} \\ & \mathcal{C}n(X) \subseteq \mathcal{F}(a), \textit{ or equivalently, } X \subseteq \mathcal{F}(a) \\ \mathcal{C}n(X) & \textit{otherwise} \end{cases}$$

extends $\mid\!\sim$ and is cumulative.

Proof: Let a be a formula. The singleton $\{a\}$ is obviously of the first type since $a \in \mathcal{C}n(a) \subseteq \mathcal{F}(a)$. By definition, then, we have $\mathcal{C}(a) = \mathcal{F}(a)$, and $\mathcal{C}$ is an extension of $\mid\!\sim$. Let us show now that $\mathcal{C}$ is cumulative. One easily sees that it is an inference operation.

Let us now show that $\mathcal{C}$ satisfies Cut and Cautious Monotonicity. We suppose that X and Y are subsets of $\mathcal{L}$ such that $Y \subseteq \mathcal{C}(X)$, and we want to prove that $\mathcal{C}(X, Y) = \mathcal{C}(X)$. If X is not of the first type, $\mathcal{C}(X) = \mathcal{C}n(X)$, therefore $Y \subseteq \mathcal{C}n(X)$ and $\mathcal{C}n(X, Y) = \mathcal{C}n(X)$. We conclude that $X \cup Y$ is not of the first type either. Hence $\mathcal{C}(X, Y) = \mathcal{C}n(X, Y) = \mathcal{C}n(X) = \mathcal{C}(X)$. If, on the contrary, X is of the first type, there exists a proposition $a \in \mathcal{C}n(X)$ such that $X \subseteq \mathcal{F}(a) = \mathcal{C}(X)$. We have $a \in \mathcal{C}n(X, Y)$ and $X \cup Y \subseteq \mathcal{F}(a)$, since, by hypothesis, $Y \subseteq \mathcal{C}(X)$. Therefore $\mathcal{C}(X, Y) = \mathcal{F}(a) = \mathcal{C}(X)$. ∎

The smallest cumulative extension described above is a very elegant technical construction but seems to fail to be a *natural* way of extending relations. For X's that are not of the first type, there is no relation between $\mathcal{C}(X)$ and the $\mathcal{F}(a)$'s for the elements a of X. In particular, one may check that, even if the relation $\mid\!\sim$ is monotonic, its smallest cumulative extension is not, in general, monotonic. For instance, take $\mathcal{L}$ to be the propositional calculus on the variables $q, p_0, \ldots, p_i, \ldots$ and define $\mid\!\sim$ by: $a \mid\!\sim b$ iff $b \in \mathcal{C}n(a, q)$. The inference relation $\mid\!\sim$ is obviously cumulative and monotonic. Let X be the set of all the p_i's. Then it is readily seen that X is not of the first type and, thus, that $\mathcal{C}(X) = \mathcal{C}n(X)$. We conclude that $q \in \mathcal{C}(p_0)$ but $q \notin \mathcal{C}(X)$.

We shall see further that the smallest cumulative extension does not preserve other important properties, such as distributivity. This motivates our presentation of another construction, the canonical extension, that provides more natural extensions.

4.3 The canonical extension

Let, again, $\mid\!\sim$ be a cumulative relation. First, we shall define an infinitary inference operation $\mathcal{C}_\sim$. The definition of $\mathcal{C}_\sim$ is quite a natural variation on the idea of compact extensions for monotonic relations. In short we shall introduce a in $\mathcal{C}_\sim(X)$ (we assume X is a theory) iff there is a proposition b in X such that, not only $b \mathrel{|}\!\sim a$, but also $b \wedge c \mathrel{|}\!\sim a$ for any c in X. Intuitively, this means that a will be inferred from X iff a is inferred from any finite subset of $\mathcal{C}n(X)$ that is big enough. Note that this is quite a conservative (small) extension. One could perhaps consider some larger extensions.

K. Schlechta [20] provided an example showing that the operation $\mathcal{C}_\sim$ is not always cumulative. This result will not be used in this paper. We provide an iterative construction that starts at $\mathcal{C}_\sim$ and provides a cumulative extension of any cumulative relation. When $\mathrel{|}\!\sim$ satisfies some additional properties, to be studied in Section 5 and further, we shall show that $\mathcal{C}_\sim$ itself is cumulative (and more). The iterative construction developed here, in addition to allowing us to build, for any cumulative relation, a cumulative extension that is nicer than the smallest cumulative extension, is of independent interest. It has been used extensively in [2, 3].

Definition 6 *Let $\mathrel{|}\!\sim$ be a cumulative relation. The operation $\mathcal{C}_\sim$ is defined in the following way: for any set $X \subseteq \mathcal{L}$,*

$$\mathcal{C}_\sim(X) \stackrel{\text{def}}{=} \{a \in \mathcal{L} \mid \exists b \in \mathcal{C}n(X) \text{ such that } b \wedge c \mathrel{|}\!\sim a,\ \forall c \in \mathcal{C}n(X)\}.$$

We shall now proceed to study the operation $\mathcal{C}_\sim$. We shall show that it is an inference operation and provides an almost cumulative extension.

Lemma 1 *The operation $\mathcal{C}_\sim$ is an inference operation.*

Proof: The definition of $\mathcal{C}_\sim(X)$ depends only on $\mathcal{C}n(X)$, and $\mathcal{C}_\sim$ is therefore right-absorbing. It is not difficult to show that $\mathcal{C}_\sim$ satisfies Inclusion: if $x \in X$, take b to be x itself and use the Reflexivity and Right Weakening properties of $\mathrel{|}\!\sim$. It is a bit more technical to show that $\mathcal{C}_\sim$ is left-absorbing. Suppose $a \in \mathcal{C}n(\mathcal{C}_\sim(X))$. By the compacity of $\mathcal{C}n$, there is a finite $A \subseteq_f \mathcal{C}_\sim(X)$ such that $a \in \mathcal{C}n(A)$. Let $a_0, a_1, \ldots, a_n$ be the elements of A. Let $b_0, b_1, \ldots, b_n$ be the corresponding elements of $\mathcal{C}n(X)$ promised by Definition 6. Let b be their conjunction:

$$b = \bigwedge_{i=0,n} b_i.$$

The proposition b is an element of $\mathcal{C}n(X)$ and, for any $i = 0, n$ and any $c \in \mathcal{C}n(X)$

$$b \wedge c \mathrel{|}\!\sim a_i.$$

Using And and Right Weakening we conclude that $b \wedge c \mathrel{|}\!\sim a$, and therefore $a \in \mathcal{C}_\sim(X)$. ∎

Lemma 2 *The operation $\mathcal{C}_\sim$ is an extension of $\mathrel{|}\!\sim$, i.e., for any proposition d, $\mathcal{C}_\sim(d) = \{b \in \mathcal{L} \mid d \mathrel{|}\!\sim b\}$.*

Proof: Let, first, a be an element of $\mathcal{C}_\sim(d)$. We shall show that $d \mathrel{|\!\sim} a$. Indeed, let $b \in \mathcal{C}n(d)$ be as in Definition 6. Since $d \in \mathcal{C}n(d)$, we have $b \wedge d \mathrel{|\!\sim} a$, and conclude by Left Equivalence.

Let, now, a be such that $d \mathrel{|\!\sim} a$. We shall show that a is an element of $\mathcal{C}_\sim(d)$. Take the b of Definition 6 to be d. For any $c \in \mathcal{C}n(d)$, we have $d \wedge c \mathrel{|\!\sim} a$ by Left Equivalence of $\mathrel{|\!\sim}$. ∎

Lemma 3 *The operation $\mathcal{C}_\sim$ satisfies Cut.*

Proof: Suppose $Y \subseteq \mathcal{C}_\sim(X)$. We want to show that $\mathcal{C}_\sim(X, Y) \subseteq \mathcal{C}_\sim(X)$. Let a be an element of the first set. There exists some $b \in \mathcal{C}n(X, Y)$ such that

$$b \wedge c \mathrel{|\!\sim} a \text{ for any } c \in \mathcal{C}n(X, Y). \tag{2}$$

By compacity of $\mathcal{C}n$, there exists a proposition $d \in \mathcal{C}n(Y)$, such that $d \rightarrow b \in \mathcal{C}n(X)$. But, by Lemma 1, $d \in \mathcal{C}_\sim(X)$, and there is some $d' \in \mathcal{C}n(X)$ such that

$$d' \wedge c \mathrel{|\!\sim} d \text{ for any } c \in \mathcal{C}n(X). \tag{3}$$

Let us consider the proposition $d' \wedge (d \rightarrow b)$. It is in $\mathcal{C}n$(X). First, by (2), using Left equivalence, for any $c \in \mathcal{C}n(X)$, $(d \rightarrow b) \wedge d \wedge d' \wedge c \mathrel{|\!\sim} a$. Secondly, by (3), for any such c, $d' \wedge (d \rightarrow b) \wedge c \mathrel{|\!\sim} d$. By Left Equivalence and Cut we conclude that $d' \wedge (d \rightarrow b) \wedge c \mathrel{|\!\sim} a$ and that $a \in \mathcal{C}_\sim(X)$. ∎

As alluded to before, $\mathcal{C}_\sim$ does not satisfy Cautious Monotonicity. Nevertheless, it satisfies some special cases of it.

Definition 7

1. *An operation $\mathcal{C}$ is said to satisfy 1-special Cautious Monotonicity iff, for any $Y \subseteq \mathcal{L}$ and any* finite *$A \subseteq_f \mathcal{L}$, $Y \subseteq \mathcal{C}(A)$ implies $\mathcal{C}(A) \subseteq \mathcal{C}(A, Y)$.*

2. *An operation $\mathcal{C}$ is said to satisfy 2-special Cautious Monotonicity iff, for any $X \subseteq \mathcal{L}$ and any* finite *$B \subseteq_f \mathcal{L}$, $B \subseteq \mathcal{C}(X)$ implies $\mathcal{C}(X) \subseteq \mathcal{C}(X, B)$.*

It is easy to check that, in part 2 of this definition, one could have considered only the case where B is a singleton. Similarly, we can define special cases of Cut.

Definition 8

1. *An operation $\mathcal{C}$ is said to satisfy 1-special Cut iff, for any $Y \subseteq \mathcal{L}$ and any* finite *$A \subseteq_f \mathcal{L}$, $Y \subseteq \mathcal{C}(A)$ implies $\mathcal{C}(A, Y) \subseteq \mathcal{C}(A)$.*

2. *An operation $\mathcal{C}$ is said to satisfy 2-special Cut iff, for any $X \subseteq \mathcal{L}$ and any* finite *$B \subseteq_f \mathcal{L}$, $B \subseteq \mathcal{C}(X)$ implies $\mathcal{C}(X, B) \subseteq \mathcal{C}(X)$.*

Lemma 4 *The operation $\mathcal{C}_\sim$ satisfies 1-special Cautious Monotonicity.*

Proof: Let A be a finite set of propositions, and let χ_A be the conjunction of all the propositions of A. By Lemmas 1 and 2,

$$\mathcal{C}_{\sim}(A) = \mathcal{C}_{\sim}(\chi_A) = \{b \in \mathcal{L} \mid \chi_A \mid\!\sim b\}.$$

Suppose that Y is a subset of $\mathcal{C}_{\sim}(A)$. We want to show that $\mathcal{C}_{\sim}(A) \subseteq \mathcal{C}_{\sim}(A, Y)$. Let a be an arbitrary element of $\mathcal{C}_{\sim}(A)$. We shall take $\chi_A \in \mathcal{C}n(A, Y)$ as the b of Definition 6 and show that $\chi_A \wedge c \mid\!\sim a$ for any c of $\mathcal{C}n(A, Y)$. Let c be such a proposition. By Lemma 1 $c \in \mathcal{C}_{\sim}(A)$, and therefore $\chi_A \mid\!\sim c$. Since $a \in \mathcal{C}_{\sim}(A)$, we have $\chi_A \mid\!\sim a$ and we conclude by Cautious Monotonicity. ∎

Lemma 5 *The operation $\mathcal{C}_{\sim}$ satisfies 2-special Cautious Monotonicity.*

Proof: Suppose that B is a finite subset of $\mathcal{C}_{\sim}(X)$, and let χ_B be its characteristic proposition. We want to show that $\mathcal{C}_{\sim}(X) \subseteq \mathcal{C}_{\sim}(X, B)$. Let a be an arbitrary element of $\mathcal{C}_{\sim}(X)$. We must prove that $a \in \mathcal{C}_{\sim}(X, B)$. By Definition 6 there is a proposition $b \in \mathcal{C}n(X)$, such that

$$b \wedge c \mid\!\sim a \text{ for any } c \in \mathcal{C}n(X).$$

By Lemma 1, $\chi_B \in \mathcal{C}_{\sim}(X)$, and there is a proposition $b' \in \mathcal{C}n(X)$, such that

$$b' \wedge c \mid\!\sim \chi_B \text{ for any } c \in \mathcal{C}n(X).$$

For every such c, we have $b \wedge b' \wedge c \mid\!\sim a$ and $b \wedge b' \wedge c \mid\!\sim \chi_B$. By Cautious Monotonicity and Left Equivalence, we conclude that

$$b \wedge b' \wedge \chi_B \wedge c \mid\!\sim a \text{ for any such } c.$$

But, for any $d \in \mathcal{C}n(X, B)$, $\chi_B \rightarrow d \in \mathcal{C}n(X)$ and

$$b \wedge b' \wedge \chi_B \wedge (\chi_B \rightarrow d) \mid\!\sim a.$$

By Left Equivalence we see that

$$b \wedge b' \wedge \chi_B \wedge d \mid\!\sim a \text{ for any } d \in \mathcal{C}n(X, B).$$

We conclude by noticing that $b \wedge b' \wedge \chi_B \in \mathcal{C}n(X, B)$. ∎

Our goal is now to transform the extension $\mathcal{C}_{\sim}$ in an iterative way, to obtain an extension that is cumulative. The following general transformation will prove interesting.

Definition 9 *Let $\mathcal{C}$ be any operation. Its transform $\mathcal{C}'$ is defined by:*

$$\mathcal{C}'(X) = \{a \in \mathcal{L} \mid \exists A \subseteq_f \mathcal{C}n(X) \text{ such that } a \in \mathcal{C}(A, Y),\ \forall Y \subseteq \mathcal{C}(X)\}.$$

A fuller study of the properties of this transformation may be found in [2] and [3]. Our first remark, that will not be used, is that the transform of any operation is compact (take Y to be empty).

Lemma 6 *If $\mathcal{C}$ is an inference operation, so is its transform $\mathcal{C}'$.*

Proof: The definition of $\mathcal{C}_{\sim}(X)$ depends only on $\mathcal{C}n(X)$, and $\mathcal{C}_{\sim}$ is therefore right-absorbing. For Inclusion, suppose $a \in X$. Take the A of Definition 9 to be the singleton $\{a\}$, and conclude by Inclusion for $\mathcal{C}$. Showing that $\mathcal{C}$' is left absorbing is done as in Lemma 1. ∎

A small effort will enable the reader to see that, if $\mathcal{C}$ is an inference operation, the set A of definition 9 may be taken as a subset of X (instead of $\mathcal{C}n(X)$). We shall not use this remark.

Lemma 7 *Let $\mathcal{C}$ be an inference operation. For any X, $\mathcal{C}'(X) \subseteq \mathcal{C}(X)$.*

Proof: We may suppose X a theory, without loss of generality. Let a be an element of $\mathcal{C}'(X)$. Let $A \subseteq_f X$ be the finite set promised by Definition 9. Since $X \subseteq \mathcal{C}(X)$, we conclude that $a \in \mathcal{C}(A, X) = \mathcal{C}(X)$. ∎

Lemma 8 *If $\mathcal{C}$ is an inference operation and satisfies 1-special Cautious Monotonicity, then $\mathcal{C}$ and $\mathcal{C}'$ agree on finite sets.*

Proof: Let $\mathcal{C}$ be an inference operation that satisfies special 1-special Cautious Monotonicity. Let A be a finite subset of $\mathcal{L}$. We must prove that $\mathcal{C}(A) = \mathcal{C}'(A)$. By Lemma 7, it is enough to show that $\mathcal{C}(A) \subseteq \mathcal{C}'(A)$. Let a be an element of the first set. We shall take A itself to be the A of Definition 9. Let Y be an arbitrary subset of $\mathcal{C}(A)$. We shall show that $a \in \mathcal{C}(A, Y)$. Indeed, since $\mathcal{C}$ satisfies 1-special Cautious Monotonicity, we deduce from $Y \subseteq \mathcal{C}(A)$ that we have $\mathcal{C}(A) \subseteq \mathcal{C}(A, Y)$. ∎

The next lemma explains our interest in the transform of an operation. It is a way of building operations that satisfy Cautious Monotonicity.

Lemma 9 *Let $\mathcal{C}$ be an inference operation.*

1. *if $\mathcal{C}$ satisfies Cut, then its transform satisfies Cautious Monotonicity.*
2. *if $\mathcal{C}$ satisfies 1-special Cut, then its transform satisfies 1-special Cautious Monotonicity.*
3. *if $\mathcal{C}$ satisfies 2-special Cut, then its transform satisfies 2-special Cautious Monotonicity.*

Proof: For the first item, let $\mathcal{C}$ be an inference operation that satisfies Cut. Let $Y \subseteq \mathcal{C}'(X)$. We must show that $\mathcal{C}'(X) \subseteq \mathcal{C}'(X, Y)$. Let a be an arbitrary element of $\mathcal{C}'(X)$. By Definition 9, there exists a finite subset A of $\mathcal{C}n(X)$ such that $a \in \mathcal{C}(A, Z)$ for any subset Z of $\mathcal{C}(X)$. But A is a finite subset of $\mathcal{C}n(X \cup Y)$, and we shall take it to be the A of Definition 9 to show that a is an element of $\mathcal{C}'(X, Y)$. Take Z to be an arbitrary subset of $\mathcal{C}(X, Y)$. We must show that $a \in \mathcal{C}(A, Z)$. But, by Lemma 7, $\mathcal{C}'(X) \subseteq \mathcal{C}(X)$, and therefore $Y \subseteq \mathcal{C}(X)$. Since $\mathcal{C}$ satisfies Cut, now, we conclude that $\mathcal{C}(X, Y) \subseteq \mathcal{C}(X)$. Therefore Z is a subset of $\mathcal{C}(X)$ and $a \in \mathcal{C}(A, Z)$. For the last two items, the proofs are exactly the same, except that certain sets are supposed to be finite. ∎

Lemma 10 *Let $\mathcal{C}$ be an inference operation that satisfies 2-special Cut.*

1. *if $\mathcal{C}$ satisfies Cautious Monotonicity, then its transform satisfies Cut.*
2. *if $\mathcal{C}$ satisfies 1-special Cautious Monotonicity, then its transform satisfies 1-special Cut.*
3. *if $\mathcal{C}$ satisfies 2-special Cautious Monotonicity, then its transform satisfies 2-special Cut.*

Proof: We shall prove the first item. The other items are proved in exactly the same way, noticing that certain sets are finite. Suppose $\mathcal{C}$ satisfies 2-special Cut and Cautious Monotonicity. Suppose $Y \subseteq \mathcal{C}'(X)$. We want to show that $\mathcal{C}'(X, Y) \subseteq \mathcal{C}'(X)$. Let a be an arbitrary element of the first set. By Definition 9 there is a finite set $A \subseteq \mathcal{C}n(X \cup Y)$ such that $a \in \mathcal{C}(A, Z)$ for any $Z \subseteq \mathcal{C}(X, Y)$. But, by Lemma 6, $\mathcal{C}'$ is an inference operation and, by hypothesis, $Y \subseteq \mathcal{C}'(X)$. We conclude that $A \subseteq \mathcal{C}'(X)$. By Lemma 6, $\chi_A \in \mathcal{C}'(X)$ and there is a finite subset $B \subseteq_f \mathcal{C}n(X)$, such that $\chi_A \in \mathcal{C}(B, Z)$ for any $Z \subseteq \mathcal{C}(X)$. We claim that B may be taken as the A of Definition 9 and shall show that $a \in \mathcal{C}'(X)$ by showing that $a \in \mathcal{C}(B, Z)$ for any $Z \subseteq \mathcal{C}(X)$. Take an arbitrary such Z. Since $\mathcal{C}$ satisfies 2-special Cut, $\mathcal{C}(B, Z, \chi_A) \subseteq \mathcal{C}(B, Z)$. But, by Lemma 7, $\mathcal{C}'(X) \subseteq \mathcal{C}(X)$ and therefore $Y \subseteq \mathcal{C}(X)$. Since $\mathcal{C}$ satisfies Cautious Monotonicity, we have $\mathcal{C}(X) \subseteq \mathcal{C}(X, Y)$. Therefore Z is a subset of this last set and $B \cup Z \subseteq \mathcal{C}(X, Y)$. We conclude that $a \in \mathcal{C}(A, B, Z) = \mathcal{C}(B, Z, \chi_A)$. Therefore, $a \in \mathcal{C}(B, Z)$. ▮

We are now ready to define the canonical extension of a cumulative relation.

Definition 10 *Let $\mid\!\sim$ be any inference relation and $\mathcal{C}_{\sim}$ the operation defined from it, as described at the beginning of Section 4.3. Let us define $\mathcal{C}_0$ to be $\mathcal{C}_{\sim}$. For any natural number $i > 0$, let us define $\mathcal{C}_i$ to be the transform of $\mathcal{C}_{i-1}$. The canonical extension of $\mid\!\sim$ is the intersection of the $\mathcal{C}_i$'s, i.e., the operation $\mathcal{C}$ such that, for any $X \subseteq L$, $\mathcal{C}(X) = \bigcap_{i \in \omega} \mathcal{C}_i(X)$.*

One may criticize our decision to call the operation defined in Definition 10 the canonical extension, since it is not always equal to its transform. It would have perhaps been wiser to call canonical extension the operation obtained, by ordinal induction, when iterating the process of taking the transform until we get some operation that is equal to its transform. All claims made below about about the canonical extension hold true for this operation too: our definition uses, as *the* canonical extension, the first (i.e. the largest) suitable operation found during our construction, i.e., the closest to $\mathcal{C}_{\sim}$.

Theorem 3 *The canonical extension of any cumulative relation is indeed an extension and is cumulative.*

Proof: Lemmas 1, 2, 3, 4 and 5 show that $\mathcal{C}_\sim$ is an inference operation that extends of $\mid\!\sim$ satisfies Cut and the two forms of special Cautious Monotonicity. Lemmas 6, 9 and 10 imply that all C_i's are inference operations, the even ones satisfy Cut and both forms of special Cautious Monotonicity, and the odd ones satisfy Cautious Monotonicity and both special forms of Cut. Lemmas 2 and 8 now imply that all C_i's are extensions of $\mid\!\sim$. We conclude that the canonical extension of a cumulative relation is indeed an extension of $\mid\!\sim$. Since the intersection of inference operations is an inference operation, it is also an inference operation. By Lemma 7, the chain of the C_i's is a descending chain, therefore the canonical extension is the intersection of the even C_i's. They all satisfy Cut, and an intersection of operations satisfying Cut satisfies Cut. We conclude that the canonical extension satisfies Cut. Similarly it satisfies Cautious Monotonicity (consider the odd indexes). ▮

We may remark that the canonical extension of a cumulative finitary operation is compact: indeed, all C_i's are compact and they all agree on finite sets.

As will be seen in Section 5, many cumulative relations $\mid\!\sim$, have a $\mathcal{C}_\sim$ that is equal to its transform. In such a case, obviously, $\mathcal{C}_\sim$ is the canonical extension. We may remark that this is also the case if $\mid\!\sim$ is any monotonic inference relation (not even cumulative). Then its $\mathcal{C}_\sim$ is its compact extension and is therefore monotonic too. Clearly, in this case, the operation $\mathcal{C}_\sim$ is equal to its transform and is the canonical extension of $\mid\!\sim$. We have, so far, provided two different ways of extending cumulative relations. In the next section we deal with the model-theory of cumulative operations.

4.4 Cumulative models

It is possible to define cumulative operations by means of models, as was done in [12] for cumulative relations. We present here an extension and a sharpening of the results of [12].

If R is a binary relation, $a \not{R} b$ will mean that the pair (a, b) does not stand in the relation R. We shall use the notions (precisely defined there) of an *asymmetric* relation, a *minimal* element and a *minimum* in a set.

Definition 11 *V is* smooth *iff $\forall t \in V$, either $\exists s$ minimal in V, such that $s \prec t$ or t is itself minimal in V.*

Definition 12 *A* cumulative *model is a triple $\langle S, l, \prec \rangle$, where S is a set, the elements of which are called states, $l : S \mapsto 2^{\mathcal{U}}$ is a function that labels every state with a non-empty set of models and $\prec$ is a binary relation on S, satisfying the* **smoothness condition** *that will be defined below in Definition 14.*

Our states correspond to what are often called worlds in the framework of Kripke semantics. Notice that the relation $\prec$ is an arbitrary binary relation. It represents the preference one may have between different states, or the degree of normality of a state, i.e., if $s \prec t$, s is preferred to or more typical than t.

Definition 13 *Let* $\langle S, l, \prec\rangle$ *be as above. If* $X \subseteq \mathcal{L}$ *is a set of formulas, we shall say that* $s \in S$ *satisfies* X *and write* $s \models\!\!\equiv X$ *iff for every model* $m \in l(s)$*, and every formula* $a \in X$*,* $m \models a$*. The set:* $\{s \mid s \in S,\ s \models\!\!\equiv X\}$ *of all states that satisfy* X *will be denoted by* $\widehat{X}$*.*

Definition 14 (smoothness condition) *A triple* $\langle S, l, \prec\rangle$ *is said to satisfy the smoothness condition iff, for any set* $X \subseteq \mathcal{L}$ *of formulas, the set* $\widehat{X}$ *is smooth.*

The reader must be cautioned that the definition given here to cumulative models is more restrictive than the one given in [12]. The smoothness property required here is stronger than the one presented there. In [12], only sets of the form $\widehat{A}$ for *finite* sets A of formulas were required to be smooth. We shall now describe how a cumulative model defines a cumulative operation.

Theorem 4 *Let* $W = \langle S, l, \prec\rangle$ *be a cumulative model and* $\mathcal{C}_W$ *the operation defined by:*

$$\mathcal{C}_W(X) = \{a \in \mathcal{L} \mid s \models\!\!\equiv a \text{ for every } s \text{ minimal in } \widehat{X}\}.$$

The operation $\mathcal{C}_W$ *is a cumulative operation.*

The proof is not difficult and may be found in [5]. We are now interested in the converse problem: given a cumulative operation $\mathcal{C}$, does there exist a cumulative model W such that $\mathcal{C} = \mathcal{C}_W$? As in the finitary case, treated in [12], the answer is positive. To see this, suppose an arbitrary cumulative operation $\mathcal{C}$ is given. Two subsets X and Y of $\mathcal{L}$ will be said to be $\mathcal{C}$-equivalent iff $\mathcal{C}(X) = \mathcal{C}(Y)$. This defines an equivalence relation on the subsets of $\mathcal{L}$. Let us denote by $[X]$ the $\mathcal{C}$-equivalence class of a set X. We put S to be the set of all such equivalence classes and define the relation $\prec$ among the elements of S by:

$$[X] \prec [Y] \text{ iff } [X] \neq [Y] \text{ and } \exists X' \subseteq \mathcal{C}(Y) \text{ such that } X' \text{ is } \mathcal{C}-\text{equivalent to } X.$$

This is a well-defined binary relation among the elements of S, i.e., the definition does not depend on the representatives used. Let us note immediately the following.

Lemma 11 *The relation* $\prec$ *is asymmetric.*

Proof: Suppose we have $[X] \prec [Y]$ and $[Y] \prec [X]$. We shall derive a contradiction. Then, there are two sets X' and Y' such that $\mathcal{C}(X') = \mathcal{C}(X)$, $\mathcal{C}(Y') = \mathcal{C}(Y)$, X' is a subset of $\mathcal{C}(Y)$ and Y' a subset of $\mathcal{C}(X)$. Now, this implies that X' is a subset of $\mathcal{C}(Y')$ and Y' a subset of $\mathcal{C}(X')$. Since $\mathcal{C}$ is cumulative, we have $\mathcal{C}(X') = \mathcal{C}(Y')$, hence $\mathcal{C}(X) = \mathcal{C}(Y)$, which contradicts $[X] \prec [Y]$. ∎

To complete the definition of the model W, we define the function l by ($\mathcal{U}$ is the set of all models of $\mathcal{L}$)

$$l([X]) = \{m \in \mathcal{U} \mid m \models \mathcal{C}(X)\}.$$

One checks easily this does not depend on the choice of the representative of [X].

Lemma 12 *A state $[X]$ is an element of $\widehat{Y}$ iff Y is a subset of $\mathcal{C}(X)$.*

Proof: Indeed $[X]$ is an element of $\widehat{Y}$ iff $l([X])$ satisfies Y. By the way l was defined, this holds iff any model m that satisfies $\mathcal{C}$(X), satisfies Y. This is equivalent to $Y \subseteq \mathcal{C}n(\mathcal{C}(X))$. We conclude by left absorption. ∎

Corollary 1 *The state $[X]$ is the minimum of $\widehat{X}$.*

Proof: From Lemma 12 and the definition of $\prec$. ∎

Corollary 2 *The state $[X]$ is minimal in $\widehat{X}$ and $\widehat{X}$ is smooth.*

Proof: Straightforward, using corollary 1 and Lemma 11. ∎

Theorem 5 *Any cumulative operation may be defined by a cumulative model in which the preference relation is an asymmetric relation such that, for every set X of formulas $\widehat{X}$ has a minimum.*

Proof: : Let indeed $\mathcal{C}$ be a cumulative operation and $W = \langle S, l, \prec \rangle$ defined as above. By corollary 2, W is a cumulative model. Let us check that $\mathcal{C}_W = \mathcal{C}$. A formula a is in $\mathcal{C}_W(X)$ iff it is satisfied by all minimal states of $\widehat{X}$. By Lemma 12 and corollary 1 this happens iff a is satisfied by $[X]$, hence iff any model m that satisfies $\mathcal{C}(X)$ satisfies also a. But this is just equivalent to $a \in \mathcal{C}n(\mathcal{C}(X)) = \mathcal{C}(X)$. ∎

By putting together our results about the extension of a cumulative relation and Theorem 5, we may now conclude that any cumulative relation may be defined by a cumulative (in the present sense) model, or more precisely is the restriction to singletons of the operation defined by some cumulative model. In other words, in [12], cumulative relations were shown to be representable by models satisfying a weak smoothness property, we showed here they may be represented also by models that enjoy the stronger smoothness property. A completeness result, weaker than Theorem 5, appears in [18].

The inference operations that may be defined by cumulative models the preference relation of which is a partial ordering may be characterized as those cumulative operations that satisfy an infinitary version of the Loop rule of [12]. For brevity's sake, we shall describe this family: the reader may consult [5].

5 Distributive operations

5.1 Definition and basic facts

In the present section we shall study a restricted family of cumulative operations. The definition of this family needs the consideration of some intricate interaction between the inference operation $\mathcal{C}$ and the operation of logical consequence $\mathcal{C}n$. We shall be able to prove interesting results about this family but shall not be able to provide a representation theorem for it. Distributive inference operations are those cumulative operations that satisfy an additional property, that is the infinitary analogue of the Or rule of [12]. In a sense, then, they are one possible infinitary analogue to the preferential relations of [12]. The results obtained there for prefential relations suggested to D. Makinson the study of this infinitary version. First results appear in [17]. In Section 6 a more restricted analogue will be proposed. We postpone until then the study of canonical extensions of preferential relations. We give here, nevertheless, those results that could help find a representation theorem for distributive operations, even though we do not present any representation result.

Definition 15 *An inference operation $\mathcal{C}$ is* distributive *iff it is cumulative and satisfies the following property, for any sets X, Y, Z of formulas:*

$$\text{(Distributivity)} \quad \mathcal{C}(Z, X) \cap \mathcal{C}(Z, Y) \subseteq \mathcal{C}(Z, \mathcal{C}n(X) \cap \mathcal{C}n(Y)) \tag{4}$$

First some comments. Since any cumulative operation satisfies right absorption, we could as well have formulated the property of Distributivity as: for any theories X, Y, Z,

$$\mathcal{C}(Z, X) \cap \mathcal{C}(Z, Y) \subseteq \mathcal{C}(Z, X \cap Y)$$

or as: for any sets X, Y, Z,

$$\mathcal{C}(Z, X) \cap \mathcal{C}(Z, Y) \subseteq \mathcal{C}(Z, X \vee Y)$$

This latest formulation is probably the most telling one: if some formula may be inferred, in the presence of assumptions Z, both from X and from Y, then it may be infered from their disjunction, in the presence of Z. There are supraclassical, monotonic operations that satisfy Cut but do not satisfy Distributivity. Therefore, contrary to Cautious Monotonicity, Distributivity is not a special case of Monotonicity. Consider, for example, classical propositional calculus and define $\mathcal{C}$ by:

$$\mathcal{C}(X) = \begin{cases} \mathcal{L} & \text{if } \mathcal{C}n(X) \neq \mathcal{C}n(\emptyset) \\ \mathcal{C}n(\emptyset) & \text{otherwise.} \end{cases}$$

The operation $\mathcal{C}$ is monotonic and cumulative, but not distributive. Notice, now, that, Distributivity looks very much like one half of the usual property defining disjunction. More precisely, it implies that disjunction behaves as half of a *disjunction for $\mathcal{C}$*, i.e. $\mathcal{C}(X, a) \cap \mathcal{C}(X, b) \subseteq \mathcal{C}(X, a \vee b)$. Distributivity is a bit stronger than this property, though, since it allows us to consider sets of formulas instead of single formulas. We shall now present a most useful tool in the study of distributive operations: any distributive operation defines an ordering on theories. This ordering may be interpreted as saying a theory is at least as normal as another one.

5.2 The ordering defined by a distributive operation

We shall define a binary relation on theories, for any inference operation $\mathcal{C}$.

Definition 16 *Let $\mathcal{C}$ be an inference operation. Let X, Y be theories. We say that $X \preceq_{\mathcal{C}} Y$ iff $X \subseteq \mathcal{C}(X \cap Y)$,*

This relation expresses that theory X is *more expected*, or less unusual than theory Y, since X is expected if the formulas that are common to both theories are true. The meaning of $\preceq_{\mathcal{C}}$ is intuitively clear, $X \preceq_{\mathcal{C}} Y$ iff $X \subseteq \mathcal{C}(X \vee Y)$ meaning that, on the premise that either X or Y holds, one infers that X holds. The following lemma expresses some basic properties of the relation $\preceq_{\mathcal{C}}$ for a *cumulative* operation $\mathcal{C}$.

Lemma 13 *Let $\mathcal{C}$ be a cumulative operation. Let X, Y be theories.*

1. *$X \subseteq Y \Rightarrow X \preceq_{\mathcal{C}} Y$, in particular, $\preceq_{\mathcal{C}}$ is reflexive,*
2. *$X \preceq_{\mathcal{C}} Y$ iff $\mathcal{C}(X) = \mathcal{C}(X \cap Y)$,*
3. *if $X \preceq_{\mathcal{C}} Y$ and X is $\mathcal{C}$-inconsistent, then Y is $\mathcal{C}$-inconsistent,*
4. *$X \preceq_{\mathcal{C}} Y$ and $Y \preceq_{\mathcal{C}} X$ imply $\mathcal{C}(X) = \mathcal{C}(Y)$,*
5. *$X \preceq_{\mathcal{C}} \mathcal{C}(X)$ and*
6. *$\mathcal{C}(X) \preceq_{\mathcal{C}} X$.*

Proof: Item 1 is proved by Inclusion. Notice that the hypothesis is $X \subseteq Y$, not the weaker $X \subseteq \mathcal{C}(Y)$. The stronger property obtained by weakening the hypothesis is the property of weak distributivity that will be defined in item 2 of Theorem 6. One direction of item 2 is proved by Cumulativity, the other one by Inclusion. For item 3, by item 2, $\mathcal{C}(X) = \mathcal{C}(X \cap Y)$. But $\mathcal{C}(X) = \mathcal{L}$. Therefore $Y \subseteq \mathcal{C}(X \cap Y)$. We conclude by Cumulativity. Item 4 follows from 2. Notice that the converse of 4 is not claimed to hold. Items 5 and 6 are proved by Supraclassicality. ∎

Our next result represents a generalization of and an improvement on a similar proof by D. Makinson and K. Schlechta for the finitary case.

Theorem 6 *Let $\mathcal{C}$ be a cumulative operation. Each one of the following two properties is equivalent to Distributivity.*

1. *For any theories X, Y, $\mathcal{C}(X) \cap \mathcal{C}(Y) \subseteq \mathcal{C}(X \cap Y)$.*
2. *For any sets of formulas X, Y, if $Y \subseteq \mathcal{C}(X)$, then $Y \preceq_{\mathcal{C}} X$.*

Proof: Property 1 is a special case of Distributivity: the case when the Z of Definition 15 is empty. Property 2 expresses the very natural (and at first sight weak) property that if a set of formulas, Y, may be inferred from X, they may be inferred from $X \vee Y$. Let us show that property 1 implies property 2. Suppose $\mathcal{C}$ is cumulative and satisfies property 1. If $Y \subseteq \mathcal{C}(X)$, we have $Y \subseteq \mathcal{C}(X) \cap \mathcal{C}(Y)$ by Inclusion. Since $\mathcal{C}(X) \cap \mathcal{C}(Y) \subseteq \mathcal{C}(X \cap Y)$, we are easily done.

We shall now show that property 2 implies property 1. Suppose $\mathcal{C}$ is a cumulative operation that satisfies property 2. To show that it satisfies property 1 we shall consider three arbitrary *theories* X, Y and Z and show that if $Z \subseteq \mathcal{C}(X) \cap \mathcal{C}(Y)$, then $Z \subseteq \mathcal{C}(X \cap Y)$. Property 1 will follow by taking $Z = \mathcal{C}(X) \cap \mathcal{C}(Y)$, this last intersection being a theory. First, since $Z \subseteq \mathcal{C}(X)$, we notice that $\mathcal{C}n(X, Z) \subseteq \mathcal{C}(X)$. Let now W be an arbitrary theory. We have $\mathcal{C}n(X, Z) \cap W \subseteq \mathcal{C}(X)$. We now apply property 2 and conclude that $\mathcal{C}n(X, Z) \cap W \subseteq \mathcal{C}(X \cap \mathcal{C}n(X, Z) \cap W) = \mathcal{C}(X \cap W)$. We shall now take W to be Y and conclude that $\mathcal{C}n(X, Z) \cap Y \subseteq \mathcal{C}(X \cap Y)$. Since $\mathcal{C}$ satisfies Cut, we conclude that $\mathcal{C}(X \cap Y, \mathcal{C}n(X, Z) \cap Y) \subseteq \mathcal{C}(X \cap Y)$. Therefore we have $\mathcal{C}(\mathcal{C}n(X, Z) \cap Y) \subseteq \mathcal{C}(X \cap Y)$. It is left to us to prove that $Z \subseteq \mathcal{C}(\mathcal{C}n(X, Z) \cap Y)$. But, similarly to what was done above, from the fact that $Z \subseteq \mathcal{C}(Y)$, we conclude that, for any theory W, we have $\mathcal{C}n(Y, Z) \cap W \subseteq \mathcal{C}(Y \cap W)$. We shall take W to be $\mathcal{C}n(X, Z)$ and are done.

We shall now show that any $\mathcal{C}$ that satisfies property 1 is distributive. We have $\mathcal{C}(Z, X) \cap \mathcal{C}(Z, Y) \subseteq \mathcal{C}(\mathcal{C}n(Z, X) \cap \mathcal{C}n(Z, Y))$. But $\mathcal{C}n(Z, X) \cap \mathcal{C}n(Z, Y) = \mathcal{C}n(Z, X \cap Y)$. One concludes by right-absorption. ∎

A corollary of Property 1 is that the operation of logical consequence, $\mathcal{C}n$ is distributive. This could have been proved more directly; it will be used in the sequel. The following will also be useful.

Lemma 14 *Let $\mathcal{C}$ be a distributive operation. For any theories X, Y, W, Z:*

1. *If $Y \subseteq \mathcal{C}(X)$, then $\mathcal{C}(Y) = \mathcal{C}(X \cap Y)$,*
2. *if $X \preceq_{\mathcal{C}} Y$ and $W \preceq_{\mathcal{C}} Z$, then $X \cap W \preceq_{\mathcal{C}} Y \cap Z$.*

Proof: For item 1, from the easy part of Theorem 6 we know that $Y \preceq_{\mathcal{C}} X$. We conclude by part 2 of Lemma 13. For item 2, we have

$$X \cap W \subseteq \mathcal{C}(X \cap Y) \cap \mathcal{C}(W \cap Z).$$

We conclude by Distributivity. ∎

The following result will be crucial in Section 6.

Theorem 7 *If the operation $\mathcal{C}$ is distributive, then the relation $\preceq_{\mathcal{C}}$ is transitive, and therefore a pre-order.*

Proof: We have seen in part 1 of Lemma 13 that $\preceq_{\mathcal{C}}$ is reflexive. Suppose $X \preceq_{\mathcal{C}} Y$ and $Y \preceq_{\mathcal{C}} Z$. We must show that $X \preceq_{\mathcal{C}} Z$. Without loss of generality we may assume that X, Y and Z are theories. From the hypotheses, by Lemma 14, part 2, we conclude that $X \cap Y \preceq_{\mathcal{C}} Y \cap Z$. Therefore, by Lemma 13, part 2 we have $\mathcal{C}(X \cap Y) = \mathcal{C}(X \cap Y \cap Z)$. But the same lemma implies $\mathcal{C}(X) = \mathcal{C}(X \cap Y)$. Similarly, from $X \preceq_{\mathcal{C}} Y$ and $Z \preceq_{\mathcal{C}} Z$, one concludes, from Lemma 14, part 2, that $X \cap Z \preceq_{\mathcal{C}} Y \cap Z$ and $\mathcal{C}(X \cap Z) = \mathcal{C}(X \cap Y \cap Z)$. Therefore $\mathcal{C}(X) = \mathcal{C}(X \cap Z)$ and $X \preceq_{\mathcal{C}} Z$. ▮

The following may help understand the meaning of the relation $\preceq_{\mathcal{C}}$, for a distributive operation $\mathcal{C}$.

Theorem 8 *Let $\mathcal{C}$ be a distributive operation. The following three properties are equivalent.*

1. $X \preceq_{\mathcal{C}} Y$
2. *there exists a set of formulas $Y' \subseteq \mathcal{C}n(Y)$ such that $X \subseteq \mathcal{C}(Y')$,*
3. *there exists a set of formulas $Y' \subseteq \mathcal{C}n(Y)$ such that $\mathcal{C}(X) = \mathcal{C}(Y')$.*

Proof: Suppose $X \preceq_{\mathcal{C}} Y$. Then $\mathcal{C}n(X) \cap \mathcal{C}n(Y)$ is a Y' suitable for property 3, by Lemma 13. It is clear that property 3 implies property 2. Suppose, now, that property 2 holds. By Lemma 13, $Y' \preceq_{\mathcal{C}} Y$. By the easy part of Theorem 6, $X \preceq_{\mathcal{C}} Y'$. We conclude, by Theorem 7, that property 1 holds. ▮

In [12], a cumulative relation that satisfies Loop but is not preferential (the term is defined in [12]) was presented. Its smallest extension provides an example of a cumulative operation that is not distributive.

5.3 Models for distributive operations

We do not know of an interesting representation theorem for distributive operations. We shall summarize what we know. It is easy to see that, any cumulative model the labeling function l of which labels each state with a singleton (i.e., a single model) defines a distributive operation, but, as was shown by K. Schlechta [21, 18], the converse does not hold. It seems that there are two interesting open questions: to find a model-theoretic characterization of distributive operations and to find a proof-theoretic characterization of those operations that may be defined by cumulative (or ordered cumulative) models in which each state is labeled by a single model.

6 Deductive Operations

6.1 Introduction and Plan

In this section we define a family of cumulative operations, deductive operations. Deductive operations are essentially those that satisfy the infinitary version of the S rule of [12]. They were shown to be the operations representable by preferential structures in [4]. Deductivity is the property named *infinite conditionalization* in [6], [18] and [21]. In 6.2, we define Deductivity and show that deductive operations are distributive. In 6.3 we show that the canonical extension of any preferential relation is a deductive operation. In Section 6.4, we study one popular way of defining inference operations, proposed by D. Poole. We show that Poole systems define a cumulative operation, finite Poole systems define an operation that coincide with the $\mathcal{C}_{\sim}$ of its restriction to singletons, Poole systems without constraints define a distributive operation and that finite Poole systems without constraints define a deductive operation that is the canonical extension of its restriction to singletons. In Section 6.5 we discuss models for deductive operations and prove a representation theorem. This result is a non-trivial variation on, and a sharpening of the representation result of [12] for preferential relations (Theorem 5.18).

6.2 Definitions and important properties

Definition 17 *An inference operation $\mathcal{C}$ is* deductive *iff it is cumulative and satisfies the following, for arbitrary $X, Y \subseteq \mathcal{L}$:*

$$\text{(Deductivity)} \quad \mathcal{C}(X, Y) \subseteq \mathcal{C}n(X, \mathcal{C}(Y)).$$

Since, as is easily checked, any operation that satisfies Deductivity satisfies Cut, we could have weakened the cumulativity requirement in Definition 17 to Cautious Monotonicity. Notice that we do not require deductive operations to be distributive, we shall soon see it follows from our definition. The property of Deductivity expresses the requirement that, if some formula a may be inferred from some assumptions Y and some additional assumptions X, then, from Y alone one could have inferred that if X holds then a must hold. It is one half of the property defining implication. Indeed, if $\mathcal{C}$ is deductive, then $b \in \mathcal{C}(X, a)$ implies $a \rightarrow b \in \mathcal{C}(X)$. Deductivity is slightly stronger than this property, though, since it encompasses also the case a is not a formula but an (infinite) set of formulas. Notice that the restriction of a deductive operation to singletons is a preferential relation, since it is a cumulative relation that satisfies the rule S and any cumulative relation that satisfies S is preferential.

Theorem 9 *Any deductive operation is distributive.*

Proof: Let $\mathcal{C}$ be deductive and let X and Y be theories. By Theorem 6, part 1, it is enough to show that $\mathcal{C}(X) \cap \mathcal{C}(Y) \subseteq \mathcal{C}(X \cap Y)$. But

$$\mathcal{C}(X) = \mathcal{C}(X, X \cap Y) \subseteq \mathcal{C}n(X, \mathcal{C}(X \cap Y))$$

by right absorption and Deductivity and similarly for Y. Therefore,

$$\mathcal{C}(X) \cap \mathcal{C}(Y) \subseteq \mathcal{C}n(X, \mathcal{C}(X \cap Y)) \cap \mathcal{C}n(Y, \mathcal{C}(X \cap Y)).$$

But we noticed that $\mathcal{C}n$ is distributive and therefore:

$$\mathcal{C}n(X, \mathcal{C}(X \cap Y)) \cap \mathcal{C}n(Y, \mathcal{C}(X \cap Y)) \subseteq \mathcal{C}n(X \cap Y, \mathcal{C}(X \cap Y)) = \mathcal{C}(X \cap Y).$$

∎

The converse of Theorem 9 does not hold. A simple example of a distributive operation that is not deductive may be found in [5]: an infinite Poole system without constraints is presented, and it is shown that the inference operation defined by this system is not deductive. In [18], it was shown that all such operations are distributive. We shall now provide a characterization of deductive operations.

Theorem 10 *Let $\mathcal{C}$ be a cumulative operation. The following three properties are equivalent.*

1. *The operation $\mathcal{C}$ is deductive,*
2. *if $Y \subseteq \mathcal{C}(X)$, then $\mathcal{C}(X) \subseteq \mathcal{C}n(X, \mathcal{C}(Y))$,*
3. *if $Y \preceq_{\mathcal{C}} X$, then $\mathcal{C}(X) \subseteq \mathcal{C}n(X, \mathcal{C}(Y))$.*

Proof: Suppose $\mathcal{C}$ is deductive and X, Y are theories. Let us show it satisfies condition 3. If $Y \preceq_{\mathcal{C}} X$, then, by Lemma 13, part 2, we have $\mathcal{C}(X \cap Y) = \mathcal{C}(Y)$. By Deductivity, though,

$$\mathcal{C}(X) = \mathcal{C}(X, X \cap Y) \subseteq \mathcal{C}n(X, \mathcal{C}(X \cap Y).$$

Therefore $\mathcal{C}(X) \subseteq \mathcal{C}n(X, \mathcal{C}(Y))$. Condition 3 implies condition 2 by Theorem 6. Let us show that property 2 implies Deductivity. Since $Y \subseteq \mathcal{C}(X, Y)$, property 2 implies $\mathcal{C}(X, Y) \subseteq \mathcal{C}n(X, Y, \mathcal{C}(Y)) = \mathcal{C}n(X, \mathcal{C}(Y))$. ∎

The following result is a key property of deductive operations.

Lemma 15 *Let $\mathcal{C}$ be a deductive operation. If $X \preceq_{\mathcal{C}} Y \preceq_{\mathcal{C}} Z$, then $Y \subseteq \mathcal{C}n(Z, \mathcal{C}(X))$.*

Proof: Without loss of generality, one may assume that X, Y and Z are theories. Suppose $X \preceq_{\mathcal{C}} Y \preceq_{\mathcal{C}} Z$. Since $\mathcal{C}$ is distributive we may apply Theorem 7 to obtain $X \preceq_{\mathcal{C}} Z$. By Lemma 14, part 2, using Distributivity again, from $X \preceq_{\mathcal{C}} Y$ and $X \preceq_{\mathcal{C}} Z$, we obtain $X \preceq_{\mathcal{C}} Y \cap Z$ and, by Lemma 13, $\mathcal{C}(X) = \mathcal{C}(X \cap Y \cap Z)$. But, by Deductivity,

$$\mathcal{C}(Y \cap Z) \subseteq \mathcal{C}n(Y \cap Z, \mathcal{C}(X \cap Y \cap Z)) \subseteq \mathcal{C}n(Z, \mathcal{C}(X)).$$

Since $Y \subseteq \mathcal{C}(Y \cap Z)$ we conclude that $Y \subseteq \mathcal{C}n(Z, \mathcal{C}(X))$. ∎

The following Theorem presents an easy result on monotonic deductive operations.

Theorem 11 *Let $\mathcal{C}$ be a deductive operation such that, for any $X \subseteq \mathcal{L}$, $\mathcal{C}(\emptyset) \subseteq \mathcal{C}(X)$. Then $\mathcal{C}$ is monotonic and compact. Moreover, $\mathcal{C}(X) = \mathcal{C}n(X, \mathcal{C}(\emptyset))$.*

Proof: Let $\mathcal{C}$ be as above. By Deductivity, $\mathcal{C}(X) \subseteq \mathcal{C}n(X, \mathcal{C}(\emptyset))$. By the other assumption, we have equality and Monotonicity is proven. Compactness is easily verified. ∎

We end up this section with a partial converse to Theorems 9 and 11.

Theorem 12 *Any distributive inference operation that is both monotonic and compact is deductive and a translation of $\mathcal{C}n$.*

Proof: Let $\mathcal{C}$ be a distributive, monotonic and compact operation. Suppose $a \in \mathcal{C}(X, Y)$. Since $\mathcal{C}$ is compact, there are $A \subseteq_f X$, $B \subseteq_f Y$ such that $a \in \mathcal{C}(A, B)$. But $\mathcal{C}(A, B) \subseteq \mathcal{C}n(A, \mathcal{C}(B))$. By Monotonicity of $\mathcal{C}n$ and $\mathcal{C}$ we see that $\mathcal{C}n(A, \mathcal{C}(B)) \subseteq \mathcal{C}n(X, \mathcal{C}(Y))$. One conclude that $\mathcal{C}(X, Y) = \mathcal{C}n(X, \mathcal{C}(Y)$ and $\mathcal{C}(X) = \mathcal{C}n(X, \mathcal{C}(\emptyset))$. ∎

6.3 Extending a preferential relation

Now, we shall study the existence of deductive extensions for an arbitrary preferential relation. Our first result is negative.

Theorem 13 *The smallest extension of a preferential relation is not always distributive.*

Proof: Let, $\mathcal{L}$ be the propositional calculus on the infinite set of variables: $q, p_1, p_2, \ldots, p_k, \ldots$. Let $a \mid\!\sim b$ iff $b \in \mathcal{F}(a) \stackrel{\text{def}}{=} \mathcal{C}n(a, q \rightarrow p_1)$. Note that $q \mid\!\sim p_1$. It is very easy to see that $\mid\!\sim$ is preferential. Let $\mathcal{C}$ be the smallest extension of $\mid\!\sim$. If Y is the set of all p_i's, we have

$$p_1 \in \mathcal{C}(Y) \cap \mathcal{C}(q).$$

We shall show that, nevertheless, p_1 is not an element of $\mathcal{C}(\mathcal{C}n(Y) \cap \mathcal{C}n(q)$. We claim, indeed, first that $\mathcal{C}n(Y) \cap \mathcal{C}n(q)$ is not of the first type. For suppose a is a proposition such that

$$a \in \mathcal{C}n(\mathcal{C}n(q) \cap \mathcal{C}n(Y)) = \mathcal{C}n(q) \cap \mathcal{C}n(Y) \subseteq \mathcal{F}(a). \tag{5}$$

Then, for any i, $p_i \vee q \in \mathcal{F}(a)$, i.e., $p_i \vee q \in \mathcal{C}n(a, q \rightarrow p_1)$. But it is not difficult to see that this implies that $p_i \vee q \in \mathcal{C}n(a)$ for any i. Now, q is not an element of $\mathcal{C}n(a)$, since q is not in $\mathcal{C}n(Y)$, so there exists a model m that satisfies a and does not satisfy q. There is a j such that p_j does not appear in a. Let m' be the model that differs from m in at most p_j and in which p_j is false. Then m' satisfies a but satisfies neither q nor p_j. A contradiction. We have shown that there is no proposition a such that (5) holds. Therefore $\mathcal{C}n(Y) \cap \mathcal{C}n(q)$ is not of the first type and

$$\mathcal{C}(\mathcal{C}n(q) \cap \mathcal{C}n(Y)) = \mathcal{C}n(\mathcal{C}n(q) \cap \mathcal{C}n(Y)) = \mathcal{C}n(q) \cap \mathcal{C}n(Y).$$

But, p_1 is not an element of $\mathcal{C}n(q)$. ∎

We may now come back to the question of extending preferential relations.

Theorem 14 *Let $\mid\!\sim$ be a preferential relation. The operation $\mathcal{C}_\sim$ is deductive and is the canonical extension of $\mid\!\sim$.*

Proof: To show that $\mathcal{C}_\sim$ is the canonical extension of $\mid\!\sim$, we shall prove that $\mathcal{C}_\sim$ is equal to its transform $\mathcal{C}_\sim{}'$. In view of Lemma 7, we have to prove that, for any X, $\mathcal{C}_\sim(X) \subseteq \mathcal{C}_\sim{}'(X)$. Take an arbitrary element a of the first set. There is a proposition $a' \in \mathcal{C}n(X)$, such that $a' \wedge b \mid\!\sim a$ for any b of $\mathcal{C}n(X)$. We must show that $a \in \mathcal{C}_\sim{}'(X)$. We shall take $\{a'\}$ to be the A of Definition 9 and show that $a \in \mathcal{C}_\sim(a', Y)$ for any $Y \subseteq \mathcal{C}_\sim(X)$. To show this, by Definition 6, it is enough to show that $a' \wedge c \mid\!\sim a$ for any c of $\mathcal{C}n(a', Y)$. Let c be such a proposition. Since $\mathcal{C}n(a', Y) \subseteq \mathcal{C}_\sim(X)$, $c \in \mathcal{C}_\sim(X)$, and therefore $a' \wedge c \in \mathcal{C}_\sim(X)$. There is some $a'' \in \mathcal{C}n(X)$ such that $a'' \wedge d \mid\!\sim a' \wedge c$ for any $d \in \mathcal{C}n(X)$. In particular $a'' \mid\!\sim a' \wedge c$, and, since $\mid\!\sim$ is preferential: $a' \wedge c \mid\!\sim a$ iff $a'' \vee (a' \wedge c) \mid\!\sim a$. But $d \stackrel{\text{def}}{=} a'' \vee (a' \wedge c) \in \mathcal{C}n(X)$ and therefore $a' \wedge d \mid\!\sim a$. Since $d \mid\!\sim a'$, we conclude $d \mid\!\sim a$.

By Theorem 3 $\mathcal{C}_\sim$ is a cumulative extension of $\mid\!\sim$. Let us show that $\mathcal{C}_\sim$ is deductive. We must show that $\mathcal{C}_\sim(X, Y) \subseteq \mathcal{C}n(X, \mathcal{C}_\sim(Y))$. Let a be an element of the first set. There is a proposition $b \in \mathcal{C}n(X, Y)$, such that $b \wedge c \mid\!\sim a$ for any $c \in \mathcal{C}n(X, Y)$. If $b \in \mathcal{C}n(X, Y)$, there is some $x \in \mathcal{C}n(X)$ such that $x \rightarrow b \in \mathcal{C}n(Y)$. To show that $a \in \mathcal{C}n(X, \mathcal{C}_\sim(Y)$ it is enough to show that $x \rightarrow a \in \mathcal{C}_\sim(Y)$. We shall show that, for any $c \in \mathcal{C}n(Y)$, $(x \rightarrow b) \wedge c \mid\!\sim x \rightarrow a$. Since $\mid\!\sim$ is preferential it is enough to show that $x \wedge b \wedge c \mid\!\sim a$. But $x \wedge c \in \mathcal{C}n(X, Y)$. ∎

6.4 Poole Systems

In [19], D. Poole defined a formalism for nonmonotonic reasoning. This formalism may be considered as a method for defining inference operations. We shall quickly recall here the Definitions of [19]. Then, we shall quote a result of D. Makinson and show that finite Poole systems (with constraints) define an operation $\mathcal{C}$ that is the canonical extension of its restriction to singletons, that Poole systems without constraints define distributive operations and that finite Poole systems without constraints define deductive operations. We shall then provide an example of a distributive operation that is not deductive.

Definition 18 *A Poole system (with constraints) is a pair $\langle D, K\rangle$ of sets of formulas. The set D is called the set of defaults. The set K is the set of constraints. The system $\langle D, K\rangle$ is said to be* finite *iff the set of defaults D is finite (the set of constraints may be infinite). It is said to be without constraints iff the set of constraints K is empty.*

Suppose such a Poole system $\langle D, K\rangle$ is given. Consider a set X of formulas. A subset $A \subseteq D$ is said to be a *basis* for X iff $\mathcal{C}n(X, A, K) \neq \mathcal{L}$ and A is a maximal subset of D with this property. We shall denote the set of all bases for X by $\mathcal{B}(X)$. The inference operation defined by $\langle D, K\rangle$ is now:

$$\mathcal{C}(X) \stackrel{\text{def}}{=} \bigcap_{A \in \mathcal{B}(X)} \mathcal{C}n(X, A). \tag{6}$$

The following is due to D. Makinson [18].

Theorem 15 *The inference operation defined by any Poole system is cumulative.*

In fact, D. Makinson has proved a slightly stronger result: such an inference operation also satisfies the infinitary version of the Loop property (see [5]). The following result suggests that our definition of the canonical extension of a finitary operation is indeed natural.

Theorem 16 *Let $\mathcal{C}$ be the inference operation defined by a* finite *Poole system. Let $\mid\!\sim$ be the restriction of $\mathcal{C}$ to singletons. The operation $\mathcal{C}$ is the canonical extension of $\mid\!\sim$, and equal to $\mathcal{C}_{\sim}$.*

Proof: In view of Lemma 7, it is enough for us to show that, for any $X \subseteq \mathcal{L}$, we have $\mathcal{C}_{\mathcal{F}}(X) \subseteq \mathcal{C}(X) \subseteq \mathcal{C}_{\mathcal{F}}'(X)$. Let us first show that $\mathcal{C}_{\mathcal{F}}(X) \subseteq \mathcal{C}(X)$. We must show that, if $a \notin \mathcal{C}(X)$, there exists no $A \subseteq_f X$ such that $a \in \mathcal{C}(A, C)$, for any $C \subseteq_f \mathcal{C}n(X)$. Suppose a is not an element of $\mathcal{C}(X)$. Then there exists a basis B for X, such that a is not in $\mathcal{C}n(X, B)$. For any $d \in D - B$, we have $\mathcal{C}n(X, B, d, K) = \mathcal{L}$, because of the maximality of B. By compacity of $\mathcal{C}n$, there exists a finite subset X' of X such that $\mathcal{C}n(X', B, d, K) = \mathcal{L}$. Let Y be the union of all the X' obtained for each d. Since D is finite, the set Y is finite. It is clear that B is a basis for Y, and therefore also a basis for $Y \cup A$ for any subset A of X. We want now to show that there is no $A \subseteq_f X$ such that $a \in \mathcal{C}(A, C)$ for any $C \subseteq_f \mathcal{C}n(X)$. Let A be an arbitrary finite subset of X. It is enough to show that $a \notin \mathcal{C}(A, Y)$. But B is a basis for $A \cup Y$, and $a \notin \mathcal{C}n(A, Y, B)$ since $a \notin \mathcal{C}n(X, B)$. We conclude that a is not in $\mathcal{C}(A, Y)$.

Let us, now, show that $\mathcal{C}(X) \subseteq \mathcal{C}_{\mathcal{F}}'(X)$. Suppose $a \in \mathcal{C}(X)$. For any basis B for X, $a \in \mathcal{C}n(X, B)$. By compactness of $\mathcal{C}n$, for any basis B for X, there is a finite subset A_B such that $a \in \mathcal{C}n(A_B, B)$. Let A_0 be the union of all those A_B. Since the set of defaults is finite, A_0 is a finite subset of X, and, for any basis B for X, $a \in \mathcal{C}n(A_0, B)$. Let, now, $\mathcal{E}$ be the set of all subsets E of D (the finite set of defaults) such that $\mathcal{C}n(X, E, K) = \mathcal{L}$. By compacity of $\mathcal{C}n$, for any $E \in \mathcal{E}$ there is a finite subset A_E of X such that $\mathcal{C}n(A, E, K) = \mathcal{L}$. Let A_1 be the union of all those A_E for $E \in \mathcal{E}$. The set A_1 is a finite union of finite sets and is therefore a finite subset of X, and, for any $E \in \mathcal{E}$, $\mathcal{C}n(A_1, E, K) = \mathcal{L}$.

Let $A \stackrel{\text{def}}{=} A_0 \cup A_1$. The set A is a finite subset of X. We must show that, for any $C \subseteq \mathcal{C}(X)$, one has $a \in \mathcal{C}(A, C)$. Let C be an arbitrary subset of $\mathcal{C}(X)$ and B an arbitrary basis for $A \cup C$. It is enough to show that $a \in \mathcal{C}n(A, C, B)$.

But $\mathcal{C}n(A_1, B, K) \neq \mathcal{L}$ and therefore $B \notin \mathcal{E}$ and $\mathcal{C}n(X, B, K) \neq \mathcal{L}$. There is, therefore, some basis B' for X such that $B \subseteq B'$. Since $C \subseteq \mathcal{C}(X)$, we have $C \subseteq \mathcal{C}n(X, B')$ and therefore $\mathcal{C}n(X, C, B', K) = \mathcal{C}n(X, B', K) \neq \mathcal{L}$ and $\mathcal{C}n(A, C, B', K) \neq \mathcal{L}$. We conclude that $B' = B$ and B is a basis for X. Therefore $a \in \mathcal{C}n(A_0, B$ and $a \in \mathcal{C}n(A, C, B)$. ∎

The result above should be compared with Theorem 14. Here, the operation $\mathcal{C}_\sim$ is not always distributive, as has been shown in [18]. Poole systems without constraints, i.e., $K = \emptyset$, however, define distributive operations.

Theorem 17 (Makinson) *The inference operation defined by any Poole system without constraints is distributive.*

Our next result shows that finite Poole systems without constraints define deductive operations.

Theorem 18 *Let $\mathcal{C}$ be the inference operation defined by a* finite *Poole system without constraints. The operation $\mathcal{C}$ is deductive.*

Proof: We know from Theorem 15 that $\mathcal{C}$ is cumulative. We must show that it satisfies Deductivity. We must show that $\mathcal{C}(X, Y) \subseteq \mathcal{C}n(X, \mathcal{C}(Y))$. Notice that, since there are only finitely many defaults, there are only finitely many bases. Since $\mathcal{L}$ is admissible, we have

$$\begin{aligned}
\mathcal{C}n(X, \mathcal{C}(Y)) &= \mathcal{C}n(X, \bigcap_{B \in \mathcal{B}(Y)} \mathcal{C}n(Y, B)) \\
&= \bigcap_{B \in \mathcal{B}(Y)} \mathcal{C}n(X, \mathcal{C}n(Y, B)) \\
&= \bigcap_{B \in \mathcal{B}(Y)} \mathcal{C}n(X, Y, B).
\end{aligned}$$

Suppose $a \in \mathcal{C}(X, Y)$. We shall show that, for any base B for Y, $a \in \mathcal{C}n(X, Y, B)$. Let B be an arbitrary basis for Y. Either $\mathcal{C}n(Y, X, B) = \mathcal{L}$ or B is a basis for $Y \cup X$. In any case, $a \in \mathcal{C}n(X, Y, B)$. ∎

As we prove now, Theorem 17 cannot be improved upon, in the sense that, even in the setting of classical propositional calculus, there is a Poole system without constraints that defines an operation that is not deductive. This provides an operation that is distributive but not deductive. A number of such operations have been proposed. D. Makinson seems to have provided the first one. The most interesting is probably the one provided by K. Schlechta and mentioned in Section 5.3; Theorem 20 shows indeed that it is not deductive. Independently of these proposals, A. Brodsky and R. Brofman offered the following.

Let $\mathcal{C}$ be classical propositional calculus and let D be the following infinite set: $p_0,\ p_1 \wedge \neg p_0,\ p_2 \wedge \neg p_1 \wedge \neg p_0,\ \ldots, p_i \wedge \neg p_{i-1} \wedge \ldots \wedge \neg p_0, \ldots$. Clearly any two different elements of D are inconsistent and therefore a basis may have at most one element. Now let $\mathcal{C}$ be the (distributive) operation defined by the Poole system $\langle D, \emptyset \rangle$. Let X be the infinite set: $p_0 \rightarrow q, p_1 \rightarrow q, \ldots, p_i \rightarrow q, \ldots$. Clearly the bases for X are exactly all the singletons of D and therefore $q \in \mathcal{C}(X)$. To show that $\mathcal{C}$ is not deductive we shall show that $q \notin \mathcal{C}n(X, \mathcal{C}(\emptyset))$. Indeed we claim that $\mathcal{C}(\emptyset)$ is the set of all tautologies, $\mathcal{C}n(\emptyset)$. Clearly the bases for $\emptyset$ are exactly all the singletons of D. Suppose $a \in \mathcal{C}n(d)$ for every default $d \in D$. Then a may be false only in propositional models in which all the p_i's are false. Since a refers to only a finite number of variables, a must be a tautology. Therefore $\mathcal{C}n(X, \mathcal{C}(\emptyset)) = \mathcal{C}n(X)$. But $q \notin \mathcal{C}n(X)$.

6.5 Full Models and Representation Theorem

In this section we shall provide a representation theorem, showing that the family of deductive operations is exactly the family of all operations that are defined by certain models. The reader remembers that in Section 5.3, we noticed that cumulative models that label states with singletons define distributive operations. But this family of models was found to be too small to define all distributive operations. Since all deductive operations are distributive, it is only natural we ask whether cumulative models that label states with singletons may define all deductive operations. We shall provide a positive answer to that question. Unfortunately, the operation defined by such a model is not always deductive. We shall define a special class of such models that is large enough to be able to define all deductive operations but small enough that all operations it defines are deductive.

Definition 19 *A cumulative ordered model $W = \langle S, l, \prec \rangle$ is said to be a full model iff*

1. *for every $s \in S$, $l(s)$ contains a single model (from now one we shall identify $l(s)$ with its unique element) and*
2. *(fullness property) for any set $X \subseteq \mathcal{L}$ and any model $m \in \mathcal{U}$ that satisfies all the formulas of $\mathcal{C}_W(X)$ there is a state s, minimal in $\widehat{X}$, such that $l(s) = m$.*

Notice that we require the relation $\prec$ to be a strict partial order, though, even without this assumption, the operation defined by a model is deductive. The second condition is difficult to check on specific models.

Theorem 19 *If $W = \langle S, l, \prec \rangle$ is a full model, the operation $\mathcal{C}_W$ is deductive.*

Proof: The reader will notice we do not use the fact that $\prec$ is a partial order. We know from Theorem 4, that $\mathcal{C}_W$ is cumulative. It is left to us to show that it satisfies Deductivity. Let $X, Y \subseteq \mathcal{L}$. Suppose $a \notin \mathcal{C}n(X, \mathcal{C}(Y))$. Then, there is a model m that satisfies X and $\mathcal{C}(Y)$ but does not satisfy a. Since m satisfies $\mathcal{C}(Y)$, by the fullness property, there is a state s minimal in $\widehat{Y}$ such that $l(s) = m$. But s is minimal in $\widehat{Y}$ and satisfies X. Therefore it is minimal in $\widehat{X \cup Y}$. This implies that $a \notin \mathcal{C}(X, Y)$. ∎

We shall now proceed to the proof of the representation theorem. From now on, $\mathcal{C}$ will be a fixed deductive operation on $\mathcal{L}$. We know from Theorem 9 that $\mathcal{C}$ is distributive. We shall build a full model that defines $\mathcal{C}$. The construction is very similar to the one appearing in [12, Section 5.3]. We shall say that a model m is normal for a set $X \subseteq \mathcal{L}$ iff m satisfies all the formulas of $\mathcal{C}(X)$. We proceed now to the construction of a full model $W = \langle S, l, \prec \rangle$ that defines $\mathcal{C}$. The set S is taken to be the set of all pairs (m, X) where X is a set of formulas and m is a normal model for X. The labelling function l is the projection on the first coordinate and the strict partial order $\prec$ is defined by: $(m, X) \prec (n, Y)$ iff $X \preceq_{\mathcal{C}} Y$ and m does not satisfy all the formulas of Y. One may notice immediately that this ensures that any state of the form (m, X) is minimal in $\widehat{X}$.

Lemma 16 *The relation $\prec$ is a strict partial order.*

Proof: The relation is clearly irreflexive, so all we have to check is that it is transitive. Suppose hence that $(m, X) \prec (n, Y)$ and $(n, Y) \prec (p, Z)$. Theorem 7 implies that $X \preceq_{\mathcal{C}} Z$. We have to show that m does not satisfy Z. But, by Lemma 15, $Y \subseteq \mathcal{C}n(Z, \mathcal{C}(X))$. The model m does not satisfy Y but satisfies $\mathcal{C}(X)$, therefore it does not satisfy Z. ∎

Lemma 17 *Let X, Y be sets of formulas and m a normal model for X. The three propositions that follow are equivalent.*

1. *The state (m, X) is minimal in $\widehat{Y}$.*
2. *The model m satisfies Y and X is not a subset of $\mathcal{C}n(Y, \mathcal{C}(\mathcal{C}n(X) \cap \mathcal{C}n(Y)))$.*
3. *The model m satisfies Y and $X \preceq_{\mathcal{C}} Y$.*

Proof: Let Z stand for $\mathcal{C}(\mathcal{C}n(X) \cap \mathcal{C}n(Y))$. Let us show that property 1 implies property 2. Suppose (m, X) is minimal in $\widehat{Y}$. Clearly m satisfies Y. If X was a subset of $Z \stackrel{\text{def}}{=} \mathcal{C}n(Y, Z)$, there would be a model n, that satisfies Z, and is therefore normal for $\mathcal{C}n(X) \cap \mathcal{C}n(Y)$, but does not satisfy X. The pair $(n, \mathcal{C}n(X) \cap \mathcal{C}n(Y))$ would therefore be a state of S and, by Lemma 13, part 1, we would have $(n, \mathcal{C}n(X) \cap \mathcal{C}n(Y)) \prec (m, X)$, a contradiction with the minimality of (m, X). Let us show now that property 2 implies property 3. If $X \subseteq \mathcal{C}n(Y, Z)$, since $\mathcal{C}n(X) \cap \mathcal{C}n(Y) \subseteq Z$, we have $X \subseteq Z$, i.e. $X \leq Y$. Let us show now that property 3 implies property 1. Suppose m satisfies Y and

$X \preceq_{\mathcal{C}} Y$. The state (m, X) is in $\widehat{Y}$. Suppose $(n, W) \prec (m, X)$. We have, by Lemma 15, $X \subseteq \mathcal{C}n(Y, \mathcal{C}(W))$. Since n is a normal model for W that does not satisfy X, it does not satisfy Y. ▮

Corollary 3 *The model W satisfies the smoothness condition. It is therefore a cumulative model.*

Proof: By Lemma 17, if (m, X) is an element of $\widehat{Y}$ and is not minimal in this set, then $X \not\subseteq \mathcal{C}n(Y, \mathcal{C}(\mathcal{C}n(X) \cap \mathcal{C}n(Y)))$. Therefore, there is a model n that is normal for $\mathcal{C}n(X) \cap \mathcal{C}n(Y)$, satisfies Y but does not satisfy X. The state $(n, \mathcal{C}n(X) \cap \mathcal{C}n(Y))$ is therefore, by Lemma 17 a minimal state of $\widehat{Y}$. But clearly, $(n, \mathcal{C}n(X) \cap \mathcal{C}n(Y)) \prec (m, X)$. ▮

Lemma 18 *The operation $\mathcal{C}_W$ is equal to $\mathcal{C}$.*

Proof: Let X be a set of formulas. Suppose that $\mathcal{C}_W(X) \not\subseteq \mathcal{C}(X)$. There would be a model m, that is normal for X and does not satisfy $\mathcal{C}_W(X)$. But the pair (m, X) would be a minimal state of $\widehat{X}$ by Lemma 17 and, by definition of $\mathcal{C}_W$ the model m satisfies all formulas of $\mathcal{C}_W(X)$. A contradiction.

Suppose now that $\mathcal{C}(X) \not\subseteq \mathcal{C}_W(X)$. There is therefore a state (m, Y) in W, minimal in $\widehat{X}$, such that m does not satisfy all formulas of $\mathcal{C}(X)$. But, by Lemma 17, $Y \leq X$. By Theorem 10, then, $\mathcal{C}(X) \subseteq \mathcal{C}n(X, \mathcal{C}(Y))$. But m satisfies X and $\mathcal{C}(Y)$, but not $\mathcal{C}(X)$. A contradiction. ▮

Corollary 4 *The model W is a full model.*

Proof: Suppose m satisfies $\mathcal{C}_W(X)$. By Lemma 18 it satisfies $\mathcal{C}(X)$ and (m, X) is a state of W, and clearly, from the definition of $\prec$ or Lemma 17, (m, X) is minimal in $\widehat{X}$. ▮

We may now conclude.

Theorem 20 *An inference operation is deductive iff it is defined by some full model.*

7 Rational Inference Operations

7.1 Introduction and Plan

The families of operations described in the previous sections provide a rich formal setting in which one may study nonmonotonic reasoning. But, even if one is of the opinion that reasonable nonmonotonic systems should implement a deductive operation, one may ask whether any deductive operation provides a bona fide reasonable nonmonotonic reasoning system. In [12] and in [16], the case was made that even deductive operations may be too wild, too nonmonotonic, and additional monotonicity requirements, termed there rationality conditions were studied. We shall study here the strongest of them in its infinitary form. We shall define Rational Monotonicity and rational operations and show that the canonical extension of a rational finitary operation is rational. We shall also provide a representation theorem for rational operations.

7.2 Rational Operations

It seems natural to be most interested in those nonmonotonic inference operations that are as monotonic as possible. One reasonable requirement is that any inference that may be drawn from a set X may also be drawn from a larger set Y, at least if this larger set is logically consistent with the set of inferences that could be drawn from X. This requirement minimizes the amount of conclusions that have to be defeated when additional information is gathered.

Definition 20 *An inference operation $\mathcal{C}$ is* rational *iff it is deductive and satisfies, for any $X, Y \subseteq \mathcal{L}$*

(Rational Monotonicity) $\mathcal{C}(X) \subseteq \mathcal{C}(X, Y)$ if Y is consistent with $\mathcal{C}(X)$.

If an inference operation is rational, its restriction to singletons is a rational relation. Notice that we require rational operations to be deductive. This decision of ours is explained by the fact we do not know of interesting results concerning cumulative operations that satisfy Rational Monotonicity but are not deductive. Notice also that Rational Monotonicity functions as a partial *other half* of the properties defining disjunction and implication. If $\mathcal{C}$ satisfies Rational Monotonicity, we have $\mathcal{C}(X, a \vee b) \subseteq \mathcal{C}(X, a) \cap \mathcal{C}(X, b)$, at least if both a and b are consistent with $\mathcal{C}(X, a \vee b)$, and also if $a \rightarrow b \in \mathcal{C}(X)$, then $b \in \mathcal{C}(X, a)$, at least if a is consistent with $\mathcal{C}(X)$. An example of [18] shows that some finite Poole systems without constraints define inference operations that are not rational (not even on singletons). The following provides a handy characterization of rational operations. It is closely related to the different equivalent rules of restricted transitivity shown to be equivalent to Rational Monotonicity (in the finitary framework) in [7]. The following theorem provides a characterization of rational operations.

Theorem 21 *An operation $\mathcal{C}$ is rational iff it is supraclassical, left-absorbing and satisfies $\mathcal{C}(X) = \mathcal{C}n(X, \mathcal{C}(Y))$, for any $X, Y \subseteq \mathcal{L}$ such that $Y \subseteq \mathcal{C}(X)$ and X is consistent with $\mathcal{C}(Y)$.*

Proof: For the *only if* part, suppose $\mathcal{C}$ is rational. It is obviously supraclassical and left-absorbing. Suppose $Y \subseteq \mathcal{C}(X)$ and X is consistent with $\mathcal{C}(Y)$. Since $\mathcal{C}$ is deductive and $Y \subseteq \mathcal{C}(X)$, by Theorem 10, $\mathcal{C}(X) \subseteq \mathcal{C}n(X, \mathcal{C}(Y))$. By Rational Monotonicity, since X is consistent with $\mathcal{C}(Y)$, we obtain $\mathcal{C}(Y) \subseteq \mathcal{C}(Y, X)$. But, since $Y \subseteq \mathcal{C}(X)$, by Cumulativity, we have $\mathcal{C}(X) = \mathcal{C}(X, Y)$ and therefore also $\mathcal{C}(Y) \subseteq \mathcal{C}(X)$ and and $\mathcal{C}n(X, \mathcal{C}(Y)) \subseteq \mathcal{C}(X)$.

For the *if* part, suppose $\mathcal{C}$ is supraclassical, left-absorbing and that, if $Y \subseteq \mathcal{C}(X)$ and X is consistent with $\mathcal{C}(Y)$, then $\mathcal{C}(X) = \mathcal{C}n(X, \mathcal{C}(Y))$. We must show that $\mathcal{C}$ is cumulative and satisfies Deductivity and Rational Monotonicity. Suppose $Y \subseteq \mathcal{C}(X)$. If $\mathcal{C}(X)$ is consistent, then, $X \cup Y$, that is a subset of $\mathcal{C}(X)$ is consistent with $\mathcal{C}(X)$ and, since $X \subseteq \mathcal{C}(X, Y)$, we have $\mathcal{C}(X, Y) = \mathcal{C}n(X, Y, \mathcal{C}(X)) = \mathcal{C}(X)$. If $\mathcal{C}(X, Y)$ is consistent, then it is consistent with X and since $X \cup Y \subseteq \mathcal{C}(X)$ we have $\mathcal{C}(X) = \mathcal{C}n(X, \mathcal{C}(X, Y)) = \mathcal{C}(X, Y)$. We are left with the case both $\mathcal{C}(X)$ and $\mathcal{C}(X, Y)$ are inconsistent. But in this case they are both equal to $\mathcal{L}$. We have shown cumulativity. Let us show Deductivity. If $X \cup Y$ is consistent with $\mathcal{C}(Y)$, since $Y \subseteq X \cap Y$, we have $\mathcal{C}(X, Y) = \mathcal{C}n(X, Y, \mathcal{C}(Y)) = \mathcal{C}n(X, \mathcal{C}(Y))$. If $X \cup Y$ is not consistent with $\mathcal{C}(Y)$, then $\mathcal{L} = \mathcal{C}n(X, Y, \mathcal{C}(Y)) = \mathcal{C}n(X, \mathcal{C}(Y))$ and obviously $\mathcal{C}(X, Y) \subseteq \mathcal{C}n(X, \mathcal{C}(Y))$. We have shown Deductivity. Let us show Rational Monotonicity. Suppose that Y is consistent with $\mathcal{C}(X)$. Then $X \cup Y$ is consistent with $\mathcal{C}(X)$ and, since $X \subseteq \mathcal{C}(X, Y)$, $\mathcal{C}(X, Y) = \mathcal{C}n(X, Y, \mathcal{C}(X))$. Therefore $\mathcal{C}(X) \subseteq \mathcal{C}(X, Y)$. ∎

7.3 The modular ordering induced by a rational operation

The relation $\preceq_{\mathcal{C}}$ is not particularly useful in the study of rational operations. Given an inference operation $\mathcal{C}$ we define a new relation on the its theories.

Definition 21 *Let $\mathcal{C}$ be an inference operation. Let X, Y be theories. We shall say that $X \prec_{\mathcal{C}} Y$ iff X is $\mathcal{C}$-consistent and Y is inconsistent with $\mathcal{C}(X \cap Y)$.*

The relation $X \prec_{\mathcal{C}} Y$ expresses that X is strictly more expected, or natural, than Y, since Y is inconsistent with what is expected on the premise that either X or Y holds. The following lemma expresses some basic properties of the relation $\prec_{\mathcal{C}}$ for a arbitrary cumulative operation $\mathcal{C}$.

Lemma 19 *Let $\mathcal{C}$ be a cumulative operation. Let X, Y be theories.*

1. *the relation $\prec_{\mathcal{C}}$ is irreflexive,*
2. *the relation $\prec_{\mathcal{C}}$ is asymmetric,*
3. *$X \prec_{\mathcal{C}} Y \Rightarrow X \preceq_{\mathcal{C}} Y$,*
4. *$X \prec_{\mathcal{C}} Y$ iff X is $\mathcal{C}$-consistent, $X \preceq_{\mathcal{C}} Y$ and Y is inconsistent with $\mathcal{C}(X)$.*

Proof: Item 1 is proved by Inclusion and right-absorption. For item 2, suppose $X \prec_{\mathcal{C}} Y$ and $Y \prec_{\mathcal{C}} X$. We shall derive a contradiction. Let $Z \stackrel{\text{def}}{=} X \cap Y$. We know that both X and Y are $\mathcal{C}$-consistent, and both X and Y are inconsistent with $\mathcal{C}(Z)$. Therefore $\mathcal{L} = \mathcal{C}n(X, \mathcal{C}(Z)) \cap \mathcal{C}n(Y, \mathcal{C}(Z))$. But the language $\mathcal{L}$ is admissible and therefore, $\mathcal{L} = \mathcal{C}n(Z, \mathcal{C}(Z)) = \mathcal{C}(Z)$, and Z is $\mathcal{C}$-inconsistent. But, by part 1 of Lemma 13, $Z \preceq_{\mathcal{C}} X$ and by part 3 of the same lemma X is

$\mathcal{C}$-inconsistent. A contradiction. For item 3, suppose $X \prec_{\mathcal{C}} Y$. Since Y is inconsistent with $Z \stackrel{\text{def}}{=} \mathcal{C}(X \cap Y)$, $X \subseteq \mathcal{L} = \mathcal{C}n(Y, Z)$. But $X \cap Y \subseteq Z$ and, $X \subseteq Z$. Item 4 follows from the previous item and item 2 of Lemma 13. ▮

One may show that the relation $\prec_{\mathcal{C}}$ is a strict partial order, but we shall show directly a stronger result, i.e. that this relation is a *modular* (to be defined) partial order if $\mathcal{C}$ is rational. This result is crucial in the proof of the representation result of Section 7.5.

Lemma 20 *If $\prec$ is a partial order on a set V, the following conditions are equivalent. A partial order satisfying them is called* modular *(this terminology is proposed in [10] as an extension of the notion of modular lattice of [11]).*

1. *for any $x, y, z \in V$ such that $x \not\prec y$, $y \not\prec x$ and $z \prec x$, then $z \prec y$,*
2. *for any $x, y, z \in V$ such that $x \prec z$, either $x \prec y$ or $y \prec z$,*
3. *the relation $\not\prec$ is transitive,*
4. *there is a totally ordered set Ω (the strict order on Ω will be denoted by $<$) and a function $r : V \mapsto \Omega$ (the ranking function) such that $s \prec t$ iff $r(s) < r(t)$.*

Lemma 21 *Any asymmetric binary relation that satisfies property 2 of Lemma 20 is transitive and therefore a modular strict partial order.*

Proof: Suppose $x \prec y \prec z$. If we had $x \not\prec z$, since we have $x \prec y$, we would have $z \prec y$, contradicting asymmetry. ▮

The relation $\prec_{\mathcal{C}}$ is the principal tool in studying rational operations. The next two lemmas state important properties of $\prec_{\mathcal{C}}$ for arbitrary deductive operations.

Lemma 22 *Let $\mathcal{C}$ be a deductive operation. For theories X, Y, Z, if $X \prec_{\mathcal{C}} Y$, then $X \cap Z \prec_{\mathcal{C}} Y$.*

Proof: Suppose $X \prec_{\mathcal{C}} Y$. We know that X is $\mathcal{C}$-consistent and that Y is inconsistent with $\mathcal{C}(X \cap Y)$. By Lemma 13, part 1, $X \cap Z \preceq_{\mathcal{C}} X$ and by part 3, $X \cap Z$ is $\mathcal{C}$-consistent. We must show that Y is inconsistent with $\mathcal{C}(X \cap Z)$. By Deductivity, we have $\mathcal{C}(X \cap Y) \subseteq \mathcal{C}n(X \cap Y, \mathcal{C}(X \cap Y \cap Z))$. But Y is inconsistent with $\mathcal{C}(X \cap Y)$. It is therefore inconsistent with $\mathcal{C}n(X \cap Y, \mathcal{C}(X \cap Y \cap Z))$. But $\mathcal{C}n(Y, \mathcal{C}n(X \cap Y, \mathcal{C}(X \cap Y \cap Z))) = \mathcal{C}n(Y, \mathcal{C}(X \cap Y \cap Z))$. We conclude that Y is inconsistent with $\mathcal{C}(X \cap Y \cap Z)$. ▮

Lemma 23 *Let $\mathcal{C}$ be a deductive operation, and X, Y be theories. If X is $\mathcal{C}$-consistent and Y is $\mathcal{C}$-inconsistent, then $X \prec_{\mathcal{C}} Y$.*

Proof: Since X is $\mathcal{C}$-consistent, we only have to show that Y is inconsistent with $\mathcal{C}(X \cap Y)$. But by Deductivity, $\mathcal{C}(Y) \subseteq \mathcal{C}n(Y, \mathcal{C}(X \cap Y))$. Since $\mathcal{C}(Y) = \mathcal{L}$, we are done. ▮

The next lemma summarizes the properties of $\prec_{\mathcal{C}}$ for rational relations.

Lemma 24 *Let $\mathcal{C}$ be a rational operation. The relation $\prec_{\mathcal{C}}$ is a modular strict partial order.*

Proof: By Lemma 19, part 2, $\prec_{\mathcal{C}}$ is asymmetric (this is where we need the admissibility assumption). By Lemma 21, it enough to show that, if $X \prec_{\mathcal{C}} Z$ and $X \not\prec_{\mathcal{C}} Y$, then $Y \prec_{\mathcal{C}} Z$. We shall suppose that X, Y and Z are theories. Suppose $X \prec_{\mathcal{C}} Z$ and $X \not\prec_{\mathcal{C}} Y$. We know that X is $\mathcal{C}$-consistent, Z is inconsistent with $\mathcal{C}(X \cap Z)$ and Y is consistent with $\mathcal{C}(X \cap Y)$. Since X is $\mathcal{C}$-consistent and $X \not\prec_{\mathcal{C}} Y$, Lemma 23 proves that Y is $\mathcal{C}$-consistent. If Z is $\mathcal{C}$-inconsistent, we conclude by Lemma 23. Assume, then, that Z is $\mathcal{C}$-consistent. We shall now prove that Z is inconsistent with $\mathcal{C}(Y \cap Z)$. We know that $\mathcal{L} = \mathcal{C}n(Z, \mathcal{C}(X \cap Z))$. By Deductivity, $\mathcal{C}(X \cap Z) \subseteq \mathcal{C}n(X \cap Z, \mathcal{C}(X \cap Y \cap Z))$. Therefore we have

$$\mathcal{L} = \mathcal{C}n(Z, \mathcal{C}n(X \cap Z, \mathcal{C}(X \cap Y \cap Z))) = \mathcal{C}n(Z, \mathcal{C}(X \cap Y \cap Z)),$$

i.e., Z is inconsistent with $\mathcal{C}(X \cap Y \cap Z)$. By Lemma 22, from $X \prec_{\mathcal{C}} Z$ we deduce $X \cap Y \prec_{\mathcal{C}} Z$. By Lemma 19, part 2, we see that $Z \not\prec_{\mathcal{C}} X \cap Y$. Since Z is $\mathcal{C}$-consistent, $X \cap Y$ is consistent with $\mathcal{C}(X \cap Y \cap Z)$. By Rational Monotonicity (this the first time we use it in this proof), we have $\mathcal{C}(X \cap Y \cap Z) \subseteq \mathcal{C}(X \cap Y)$. But Y is consistent with $\mathcal{C}(X \cap Y)$, and we conclude that Y is consistent with $\mathcal{C}(X \cap Y \cap Z)$. Therefore $Y \cap Z$ is consistent with $\mathcal{C}(X \cap Y \cap Z)$. By Rational Monotonicity, now, we have $\mathcal{C}(X \cap Y \cap Z) \subseteq \mathcal{C}(Y \cap Z)$. But Z is inconsistent with $\mathcal{C}(X \cap Y \cap Z)$, it is therefore inconsistent with $\mathcal{C}(Y \cap Z)$. ▮

7.4 Canonical Extensions of Rational Operations

We shall now show that canonical extensions preserve Rational Monotonicity. The first proof of this result was due to David Makinson.

Theorem 22 *Let $\mid\!\sim$ be a rational relation. The canonical extension of $\mid\!\sim$ is rational and equal to $\mathcal{C}_{\sim}$.*

Proof: In view of Theorem 14, we only have to check that the operation $\mathcal{C}_{\sim}$ satisfies Rational Monotonicity. Suppose, indeed that Y is consistent with $\mathcal{C}_{\sim}(X)$. We shall show that $\mathcal{C}_{\sim}(X) \subseteq \mathcal{C}_{\sim}(X, Y)$. Let a be an arbitrary element of $\mathcal{C}_{\sim}(X)$. There is some $b \in \mathcal{C}n(X)$ such that $b \wedge c \mid\!\sim a$ for every $c \in \mathcal{C}n(X)$. But $b \in \mathcal{C}n(X, Y)$ and we shall show that $b \wedge d \mid\!\sim a$ for every $d \in \mathcal{C}n(X, Y)$. Take an arbitrary such d. Since Y is consistent with $\mathcal{C}_{\sim}(X)$, so is $\mathcal{C}n(X, Y)$ and $\neg d \notin \mathcal{C}_{\sim}(X)$. Therefore, there is a proposition $e \in \mathcal{C}n(X)$ such that $b \wedge e \not\mid\!\sim \neg d$. Therefore $b \wedge (d \vee e) \not\mid\!\sim \neg d$. But $d \vee e \in \mathcal{C}n(X)$ and $b \wedge (d \vee e) \mid\!\sim a$. Since $\mid\!\sim$ is rational, we have $b \wedge d \mid\!\sim a$. ▮

7.5 Modular Models and Rational Operations

We shall now describe a family of full models that corresponds exactly to rational operations. The representation result is the infinitary version of the corresponding result of [16].

Definition 22 *A full model* $W \stackrel{\text{def}}{=} \langle S, l, \prec \rangle$ *for which the strict partial order* $\prec$ *is modular is said to be a modular model.*

Notice that our modular models are similar to the ranked models of [16] but differ from them in two respects: the smoothness property we require from modular models is stronger and we require them to be full.

Theorem 23 *If* W *is a modular model, the inference operation* $\mathcal{C}_W$ *it defines is rational.*

Proof: By Theorem 19 it is enough to show that $\mathcal{C}_W$ satisfies Rational Monotonicity. Suppose W is a modular model. We shall use the notations of Definition 22. Suppose also that Y is consistent with $\mathcal{C}_W(X)$. Since W is full, we conclude that there is a minimal element $s \in S$ of $\widehat{X}$ that satisfies Y. Let t be any minimal element of $\widehat{X \cup Y}$. Clearly t is an element of $\widehat{X}$. We claim it is a minimal element of $\widehat{X}$. Indeed, if it is not the case, by the smoothness condition, there is a state $v \prec t$ that is minimal in $\widehat{X}$. But, since s is minimal in $\widehat{X}$, $v \not\prec s$ and, by modularity, $s \prec t$. But s is in $\widehat{X \cup Y}$ and t is minimal in this set. A contradiction. We have shown that any minimal element of $\widehat{X \cup Y}$ is a minimal element of $\widehat{X}$. Therefore $\mathcal{C}_W(X) \subseteq \mathcal{C}_W(X, Y)$. ▮

We want, now, to prove the converse of Theorem 23. The method of proof we propose is an improvement on that of [16]. The representation theorem that results is also a strengthening of the corresponding result there (see our comments at the end of Section 4.4). Suppose a rational operation $\mathcal{C}$ is given. Notice that, by Theorem 9, $\mathcal{C}$ is distributive. We shall define the structure W to be the triple $\langle S, l, \prec \rangle$, where

- S is the set of all pairs (m, X) where m is a normal model for X, i.e. $m \models \mathcal{C}(X)$, and m does not satisfy all the formulas of $\mathcal{L}$,
- l is the projection on the first coordinate, i.e. $l(m, X) = m$ and
- the relation $\prec$ is defined by: $(m, X) \prec (n, Y)$ iff $X \prec_{\mathcal{C}} Y$.

Notice that, if (m, X) is a state, then X is $\mathcal{C}$-consistent, since m is normal for X but does not satisfy $\mathcal{L}$. Our first task is to show that W is a modular model: we must show it satisfies the smoothness condition, the fullness condition and that $\prec$ is a modular strict partial order. Then, we shall show that it defines the operation $\mathcal{C}$.

Lemma 25 *The binary relation* $\prec$ *on* S *is a modular strict partial order.*

Proof: Obvious from Lemma 24. ∎

Lemma 26 *1. If Y is $\mathcal{C}$-consistent, a state (m, X) of S is minimal in $\widehat{Y}$ iff $m \models Y$ and $Y \not\prec_{\mathcal{C}} X$.*

2. If Y is $\mathcal{C}$-inconsistent, the set $\widehat{Y}$ is empty.

Proof: Let, first, Y be $\mathcal{C}$-consistent. There is a model n that is normal for Y. The pair (n, Y) is a state of S. Suppose first that (m, X) is minimal in $\widehat{Y}$. By the definition of l, m satisfies Y. If we had $Y \prec_{\mathcal{C}} X$, we would have $(n, Y) \prec (m, X)$ for a state (n, Y) that satisfies Y, contradicting the minimality of (m, X). Therefore $Y \not\prec_{\mathcal{C}} X$.

Suppose now that $(m, X) \in \widehat{Y}$ and that $Y \not\prec_{\mathcal{C}} X$. If there was a state $(n, Z) \prec (m, X)$ such that $n \models Y$, we would have $Z \prec_{\mathcal{C}} X$, and n would satisfy both $\mathcal{C}(Z)$ and Y and Y would be consistent with $\mathcal{C}(Z)$ and therefore we would have $Z \not\prec_{\mathcal{C}} Y$. But, we would have $Z \not\prec_{\mathcal{C}} Y \not\prec_{\mathcal{C}} X$ and, since $\prec_{\mathcal{C}}$ is modular, by Lemma 24, we would have, by item 3 of Lemma 20, $Z \not\prec_{\mathcal{C}} X$. A contradiction.

Let, now, Y be $\mathcal{C}$-inconsistent. Suppose $(m, X) \in \widehat{Y}$: m satisfies $\mathcal{C}(X)$ and Y. Since $\mathcal{L} = \mathcal{C}(Y)$, we have, by Cautious Monotonicity, $\mathcal{L} = \mathcal{C}(Y, X)$ and, by Deductivity, we obtain $\mathcal{L} = \mathcal{C}n(Y, \mathcal{C}(X))$. A contradiction. ∎

Lemma 27 *The structure W satisfies the smoothness condition.*

Proof: Let (m, X) be in $\widehat{Y}$. Either $Y \not\prec_{\mathcal{C}} X$ and (m, X) is minimal in $\widehat{Y}$, by Lemma 26, or $Y \prec_{\mathcal{C}} X$. In such a case, Y is $\mathcal{C}$-consistent and there is a normal model n for Y and we have $(n, Y) \prec (m, X)$. But (n, Y) is, by Lemma 26, a minimal state in $\widehat{Y}$. ∎

Lemma 28 *The operations $\mathcal{C}$ and $\mathcal{C}_W$ coincide.*

Proof: If $a \in \mathcal{C}_W(Y)$, a is satisfied in every model m that is normal for Y, since for such a model (m, Y) is a minimal state of $\widehat{Y}$. Therefore, we have $a \in \mathcal{C}n(\mathcal{C}(Y)) = \mathcal{C}(Y)$. Suppose, now, that $a \in \mathcal{C}(Y)$. Let (n, X) be minimal in $\widehat{Y}$. By Lemma 26 $Y \not\prec_{\mathcal{C}} X$. If Y is $\mathcal{C}$-inconsistent, then $\widehat{Y}$ is empty by Lemma 26 and $\mathcal{C}_W(Y) = \mathcal{L}$. If Y is $\mathcal{C}$-consistent, then, X is consistent with $\mathcal{C}(\mathcal{C}n(X) \cap \mathcal{C}n(Y))$. Therefore, by Rational Monotonicity, $\mathcal{C}(\mathcal{C}n(X) \cap \mathcal{C}n(Y)) \subseteq \mathcal{C}(X)$ But, by Deductivity, $\mathcal{C}(Y) \subseteq \mathcal{C}n(Y, \mathcal{C}(\mathcal{C}n(X) \cap \mathcal{C}n(Y)))$. We conclude that $\mathcal{C}(Y) \subseteq \mathcal{C}n(Y, \mathcal{C}(X))$. Since n satisfies both $\mathcal{C}(X)$ and Y it satisfies a. We conclude that $a \in \mathcal{C}_W(Y)$. ∎

Lemma 29 *The structure W satisfies the fullness property.*

Proof: If m satisfies all the formulas of $\mathcal{C}_W(X)$, by Lemma 28, it is normal for X. If m does not satisfy all the formulas of $\mathcal{L}$, the pair (m, X) is a state in W and it is minimal in $\widehat{X}$. ∎

We may now conclude.

Theorem 24 *An inference operation $\mathcal{C}$ is rational iff it is defined by some modular model.*

Proof: By Theorem 23 and the construction just above. ∎

8 Conclusions and Further Work

We have presented a number of families of nonmonotonic inference operations. We have studied the question of extending nonmonotonic inference relations and obtained representation results. Most of the insightful ideas about infinitary operations have come from the study of finitary relations. The first benefit gained from such a study of infinitary operations is a streamlining of the results and the proofs. But, now, time is perhaps ripe for the study of infinitary operations to suggest new ideas concerning finitary relations. The following question seems worth a look: what are the relations $\mid\!\sim$ for which $\mathcal{C}_{\sim}$ is equal to its transform? All preferential relations satisfy this property, but so do all relations defined by finite Poole systems, and those are not always preferential.

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References

[1] Jürgen Dix and David Makinson. The relationship between klm and mak models for nonmonotonic inference operations. *Journal of Logic, Language and Information*, 1(2):131–140, 1992.

[2] Michael Freund. Supracompact inference operations. In J. Dix, K. P. Jantke, and P.H. Schmitt, editors, *Nonmonotonic and Inductive Logic, First International Workshop, Karlsruhe Germany*, pages 59–73. Springer Verlag, December 1990. Lecture Notes in Artificial Intelligence, Vol. 543.

[3] Michael Freund. Supracompact inference operations. Submitted, January 1992.

[4] Michael Freund and Daniel Lehmann. Deductive inference operations. In J. van Eijck, editor, *Proceedings of JELIA 1990 European Workshop on Logical Methods in Artificial Intelligence Lecture Notes in AI Vol. 478*, pages 227–233, Amsterdam, the Netherlands, September 1990. Springer Verlag.

[5] Michael Freund and Daniel Lehmann. Nonmonotonic inference operations. *Bulletin of the IGPL*, 1(1):23–68, July 1993.

[6] Michael Freund, Daniel Lehmann, and David Makinson. Canonical extensions to the infinite case of finitary nonmonotonic inference operations. In *Workshop on Nomonotonic Reasoning*, pages 133–138, Sankt Augustin, FRG, December 1989. Arbeitspapiere der GMD no. 443.

[7] Michael Freund, Daniel Lehmann, and Paul Morris. Rationality, transitivity, and contraposition. *Artificial Intelligence*, 52(2):191–203, December 1991. Research Note.

[8] Dov M. Gabbay. Theoretical foundations for non-monotonic reasoning in expert systems. In Krzysztof R. Apt, editor, *Proc. of the NATO Advanced Study Institute on Logics and Models of Concurrent Systems*, pages 439–457, La Colle-sur-Loup, France, October 1985. Springer-Verlag.

[9] Gerhard Gentzen. *The Collected Papers of Gerhard Gentzen, edited by M. E. Szabo.* North Holland, Amsterdam, 1969.

[10] Matthew L. Ginsberg. Counterfactuals. *Artificial Intelligence*, 30:35–79, 1986.

[11] George Grätzer. *Lattice Theory.* W. H. Freeman, San Francisco, 1971.

[12] Sarit Kraus, Daniel Lehmann, and Menachem Magidor. Nonmonotonic reasoning, preferential models and cumulative logics. *Artificial Intelligence*, 44(1–2):167–207, July 1990.

[13] Daniel Lehmann. What does a conditional knowledge base entail? In Ron Brachman and Hector Levesque, editors, *Proceedings of the First International Conference on Principles of Knowledge Representation and Reasoning*, Toronto, Canada, May 1989. Morgan Kaufmann.

[14] Daniel Lehmann. Plausibility logic. In Egon Boerger, Gerhard Jäger, Hans Kleine Buening, and Michael M. Richter, editors, *Proceedings of Computer Science Logic '91*, volume 626, pages 227–241, Berne, Switzerland, October 1991. Lecture Notes in Computer Science, Springer Verlag.

[15] Daniel Lehmann and Menachem Magidor. Preferential logics: the predicate calculus case. In Rohit Parikh, editor, *Proceedings of the Third Conference on Theoretical Aspects of Reasoning About Knowledge*, pages 57–72, Monterey, California, March 1990. Morgan Kaufmann.

[16] Daniel Lehmann and Menachem Magidor. What does a conditional knowledge base entail? *Artificial Intelligence*, 55(1):1–60, May 1992.

[17] David Makinson. General theory of cumulative inference. In M. Reinfrank, J. de Kleer, M. L. Ginsberg, and E. Sandewall, editors, *Proceedings of the Second International Workshop on Non-Monotonic Reasoning*, pages 1–18, Grassau, Germany, June 1988. Springer Verlag. Volume 346, Lecture Notes in Artificial Intelligence.

[18] David Makinson. General patterns in nonmonotonic reasoning. In D. M. Gabbay, C. J. Hogger, and J. A. Robinson, editors, *Handbook of Logic in Artificial Intelligence and Logic Programming Vol. 2, Nonmonotonic and Uncertain Reasoning*. Oxford University Press, due 1993. in preparation.

[19] David Poole. A logical framework for default reasoning. *Artificial Intelligence*, 36:27–47, 1988.

[20] Karl Schlechta. Results on infinite extensions. *Journal of Applied Non-Classical Logics*, 1(1):65–72, 1991.

[21] Karl Schlechta. Some results on classical preferential models. *Journal of Logic and Computation*, 2(6), 1992. in print.

[22] Alfred Tarski. *Logic, Semantics, Metamathematics. Papers from 1923–1938.* Clarendon Press, Oxford, 1956.