The logical meaning of Expansion

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April 12, 2000

Abstract

Some properties of choice functions, considered by researchers in Social Choice, have been shown [5] to correspond to logical properties of nonmonotonic consequence operations. Another one of those properties, Expansion, is shown to correspond to a logical property of nonmonotonic consequence relations that is the *pure*, i.e., not involving connectives, version of a previously known weak rationality condition. The assumption that the union of two definable sets of models is definable is needed for the soundness part of the result.

1 Introduction

1.1 Background

The Social Choice community considers the following setting: each individual has personal preferences over a set of possible outcomes. The society, given a subset of those, the feasible outcomes, must come up with a the subset of those feasible outcomes that are acceptable socially, in view of the individual

*This work was done while the author was on sabbatical leave at the Robotics Laboratory, Stanford University

preferences. Different methods of social decision result in different functions from sets of feasible outcomes to sets of acceptable outcomes. Social Choice investigates the relations between those different methods for social decision and the choice functions they determine. In a previous work [5] I have shown that a set of three properties of choice functions, previously studied by researchers in the theory of social preferences: Contraction, Coherence and Local Monotonicity corresponds exactly to an important family of nonmonotonic consequence operations, characterized by five properties: Inclusion, Idempotence, Cautious Monotonicity, Conditional Monotonicity and Threshold Monotonicity. The choice functions that, in addition, satisfy a condition proposed by Arrow correspond exactly to the operations that, in addition, satisfy Rational Monotonicity. In this work, an additional property of choice functions is considered and characterized in logical terms.

1.2 Recall

To make this work self-contained notations and definition from [5] are recalled here. Let us assume a non-empty set (language) $\mathcal{L}$ and a function $\mathcal{C} : 2^{\mathcal{L}} \longrightarrow 2^{\mathcal{L}}$ are given. Nothing is assumed about the language. Assume $\mathcal{M}$ is a set (of models), about which no assumption is made, and $\models \subseteq \mathcal{M} \times \mathcal{L}$ is a (satisfaction) binary relation. For any set $A \subseteq \mathcal{L}$, we shall denote by $\widehat{A}$ or by $\mathrm{Mod}(A)$ the set of all models that satisfy all elements of A:

$$\widehat{A} = \mathrm{Mod}(A) = \{x \in \mathcal{M} \mid x \models a, \ \forall a \in A\}.$$

For typographical reasons we shall use both notations, sometimes even in the same formula. For any set of models $X \subseteq \mathcal{M}$, we shall denote by $\overline{X}$ the set of all formulas that are satisfied in all elements of X:

$$\overline{X} = \{a \in \mathcal{L} \mid x \models a, \forall x \in X\}.$$

The following are easily proven, for any $A, B \subseteq \mathcal{L}$, $X, Y \subseteq \mathcal{M}$: they amount to the fact that the operations $X \mapsto \overline{X}$ and $A \mapsto \widehat{A}$ form a Galois connection.

$$A \subseteq \overline{\widehat{A}} \quad , \quad X \subseteq \widehat{\overline{X}}$$

$$\widehat{A \cup B} = \widehat{A} \cap \widehat{B} \quad , \quad \overline{X \cup Y} = \overline{X} \cap \overline{Y}$$

$$A \subseteq B \Rightarrow \widehat{B} \subseteq \widehat{A} \quad , \quad X \subseteq Y \Rightarrow \overline{Y} \subseteq \overline{X}$$

$$A \subseteq B \Rightarrow \overline{\widehat{A}} \subseteq \overline{\widehat{B}} \quad , \quad X \subseteq Y \Rightarrow \widehat{\overline{X}} \subseteq \widehat{\overline{Y}}$$

$$\widehat{A} = \widehat{\overline{\widehat{A}}} \quad , \quad \overline{X} = \overline{\widehat{\overline{X}}}$$

Definition 1 *A set X of models is said to be definable iff one of the two following equivalent conditions holds:*

1. *$\exists A \subseteq \mathcal{L}$ such that $X = \widehat{A}$, or*
2. *$X = \widehat{\overline{X}}$.*

The set of all definable subsets of X will be denoted by D_X.

Intuitively, a set of models is definable iff it is the set of all models satisfying some set of formulas.
We consider choice functions $f : 2^{\mathcal{M}} \longrightarrow 2^{\mathcal{M}}$ that choose, for a definable set X of models, the set of *best*, *most typical*, *most important*, or *preferred* models for the set X. We shall consider only choice functions that send definable sets to definable sets, i.e., we assume:

Definability Preservation $\forall A \subseteq \mathcal{L},\ f(\widehat{A}) = \widehat{\overline{f(\widehat{A})}}$

Then we define the function $\mathcal{C}$ by

$$\mathcal{C}(A) = \overline{f(\widehat{A})}. \tag{1}$$

In [5] the following properties, to be understood for every definable sets X, Y have been considered.

Contraction $f(X) \subseteq X$

Coherence $X \subseteq Y \Rightarrow X \cap f(Y) \subseteq f(X)$

Local Monotonicity $f(X) \subseteq Y \subseteq X \Rightarrow f(Y) \subseteq f(X)$

Arrow $X \subseteq Y,\ X \cap f(Y) \neq \emptyset \ \Rightarrow\ f(X) \subseteq f(Y)$

Turning now to (nonmonotonic) operations $\mathcal{C}$ we use the following, customary, notations: $\mathcal{C}(A, B)$ for $\mathcal{C}(A \cup B)$, $\mathcal{C}(a)$ for $\mathcal{C}(\{a\})$ and $\mathcal{C}(A, a)$ for $\mathcal{C}(A \cup \{a\})$. The following properties have been studied in [5].

Inclusion $\forall A \subseteq \mathcal{L}\ A \subseteq \mathcal{C}(A)$,

$$\textbf{Idempotence} \quad \forall A \subseteq \mathcal{L}\ \mathcal{C}(\mathcal{C}(A)) = \mathcal{C}(A).$$

$$\textbf{Cautious Monotonicity} \quad \forall A, B \subseteq \mathcal{L}\ A \subseteq B \subseteq \mathcal{C}(A) \Rightarrow \mathcal{C}(A) \subseteq \mathcal{C}(B)$$

$$\textbf{Conditional Monotonicity} \quad \forall A, B \subseteq \mathcal{L},\ \mathcal{C}(A, B) \subseteq \mathcal{C}(\mathcal{C}(A), B)$$

$$\textbf{Threshold Monotonicity} \quad \mathcal{C}(A) \subseteq B \subseteq C\ \Rightarrow\ \mathcal{C}(B) \subseteq \mathcal{C}(C)$$

The following have been proved in [5].

Theorem 1 *Suppose we are given a language $\mathcal{L}$ and a function $\mathcal{C} : 2^{\mathcal{L}} \longrightarrow 2^{\mathcal{L}}$. Then, the following two conditions are equivalent:*

1. *$\mathcal{C}$ satisfies Inclusion, Idempotence, Cautious Monotonicity, Conditional Monotonicity and Threshold Monotonicity,*
2. *there exists a set $\mathcal{M}$ (of models), a satisfaction relation $\models \subseteq \mathcal{M} \times \mathcal{L}$ and a definability-preserving choice function $f : D_{\mathcal{M}} \longrightarrow D_{\mathcal{M}}$ satisfying Contraction, Coherence and Local Monotonicity such that $\mathcal{C}(A) = \overline{f(\widehat{A})}$.*

Theorem 2 *Suppose we are given a language $\mathcal{L}$ and a function $\mathcal{C} : 2^{\mathcal{L}} \longrightarrow 2^{\mathcal{L}}$. Then, the following two conditions are equivalent:*

1. *$\mathcal{C}$ satisfies Inclusion, Idempotence, Cautious Monotonicity, Conditional Monotonicity, Threshold Monotonicity and*

   $$\textbf{Rational Monotonicity} \quad \mathcal{C}(\mathcal{C}(A), B) \neq \mathcal{L}\ \Rightarrow\ \mathcal{C}(A) \subseteq \mathcal{C}(A, B),$$

2. *there exists a set $\mathcal{M}$ (of models), a satisfaction relation $\models \subseteq \mathcal{M} \times \mathcal{L}$ and a definability-preserving choice function $f : \mathcal{M} \longrightarrow \mathcal{M}$ satisfying Contraction, Coherence, Local Monotonicity and Arrow such that $\mathcal{C}(A) = \overline{f(\widehat{A})}$.*

1.3 Expansion

Another property has been widely considered in Social Choice:

$$\textbf{Expansion} \quad f(X) \cap f(Y) \subseteq f(X \cup Y).$$

This says that an outcome that is acceptable if the society has to choose from a set X and also acceptable if it had to choose from a set Y (in particular it is both in X and in Y) is also acceptable if one has to choose in the union of X and Y. It does not follow from Contraction, Coherence and Local Monotonicity. Under the simplifying assumption that all sets are definable, Expansion follows from Contraction and Arrow.

It is therefore a reasonable assumption that Expansion (in the presence of Contraction, Coherence and Local Monotonicity), corresponds exactly to some property of the consequence operation that is weaker than Rational Monotonicity. Many such properties have been studied in [4, 7, 3, 1, 2].

It shall be shown that, indeed, Expansion is, essentially, the exact equivalent of one of those properties: the property found by Satoh [7] and Freund [1] to characterize the existence of injective preferential models in a finitary framework. The property is typically written: if $a \vee b \mid\!\sim c$, then there exists a' and b' such that $a \mid\!\sim a'$, $b \mid\!\sim b'$ and $a' \wedge b' \models c$, where $\models$ is logical implication. The same property may be written:

$$\mathcal{C}(\mathcal{C}n(A) \cap \mathcal{C}n(B)) \subseteq \mathcal{C}n(\mathcal{C}(A), \mathcal{C}(B)). \tag{2}$$

Two important remarks must be made immediately. The property above assumes the existence of an underlying monotonic consequence operation ($\models$, $\mathcal{C}n$), of which $\mathcal{C}$ is an extension. We do not assume the existence of such an operation and therefore, shall use, in place of $\mathcal{C}n(A)$, the closest operation that can be defined in terms of $\mathcal{C}$, i.e., $\bigcap_{F \supseteq A} \mathcal{C}(F)$. The finitary form also assumes the existence of a disjunction in the language. We shall need to make a similar assumption: we shall assume that the union of two definable sets is definable. The fact that such an assumption is needed indicates that Expansion is not a property that is as natural as the other properties considered. The question of whether the result holds without this assumption is open.

The property we propose to consider is:

$$(\mathbf{E}) \quad \mathcal{C}(\bigcap_{F \supseteq A \text{ or } F \supseteq B} \mathcal{C}(F)) \subseteq \mathcal{C}(\mathcal{C}(A), \mathcal{C}(B)).$$

Notice that, due to the Threshold Monotonicity property, the right hand side is equal to $\bigcap_{F \supseteq \mathcal{C}(A) \cup \mathcal{C}(B)} \mathcal{C}(F)$, the exact translation of the right hand side of Equation 2.

Leaving the exact formulation of the result for later, let us examine the situation.

The completeness direction seems easier than the soundness one. Assume $\mathcal{C} : 2^{\mathcal{L}} \longrightarrow 2^{\mathcal{L}}$ satisfies Inclusion, Idempotence, Cautious Monotonicity, Conditional Monotonicity, Threshold Monotonicity and property (E). As in the proof of the main representation result of previous work, we take $\mathcal{M}$ to be the set of all theories (sets of formulas closed under $\mathcal{C}$) and define the satisfaction relation by: $T \models a$ iff $a \in T$. We define f by: $f(X) = X \cap \widehat{\mathcal{C}(\overline{X})}$. We have shown in the previous proof that f preserves definability and satisfies Contraction, Coherence and Local Monotonicity and that $\mathcal{C}(A) = \overline{f(\widehat{A})}$. We easily notice that, for any set X,

$$f(X) \subseteq f(\widehat{\overline{X}}). \tag{3}$$

We already noticed in [5] that:

$$\overline{\widehat{A}} = \bigcap_{B \supseteq A} \mathcal{C}(B) \tag{4}$$

and that

$$f(\widehat{A}) = \widehat{\mathcal{C}(A)} \tag{5}$$

Notice, first, that, by (4), for any A, B:

$$\mathcal{C}(\bigcap_{F \supseteq A \text{ or } F \supseteq B} \mathcal{C}(F)) = \mathcal{C}(\overline{\widehat{A}} \cap \overline{\widehat{B}})$$

and that, by Threshold Monotonicity and by (4), for any A, B,

$$\mathcal{C}(\mathcal{C}(A), \mathcal{C}(B)) = \bigcap_{F \supseteq \mathcal{C}(A) \cup \mathcal{C}(B)} \mathcal{C}(F) = \overline{\mathrm{Mod}(\mathcal{C}(A) \cup \mathcal{C}(B))}.$$

Property (E) then becomes:

$$(\mathbf{E}') \qquad \mathcal{C}(\overline{\widehat{A}} \cap \overline{\widehat{B}}) \subseteq \overline{\mathrm{Mod}(\mathcal{C}(A) \cup \mathcal{C}(B))}.$$

Therefore we have:

$$(\mathbf{E}'') \qquad \mathrm{Mod}(\mathcal{C}(A) \cup \mathcal{C}(B)) = \mathrm{Mod}(\overline{\mathrm{Mod}(\mathcal{C}(A) \cup \mathcal{C}(B))}) \subseteq \mathrm{Mod}(\mathcal{C}(\overline{\widehat{A}} \cap \overline{\widehat{B}})).$$

We want to show that Expansion holds true. Let X and Y be arbitrary sets of models and let us define $A = \overline{X}$ and $B = \overline{Y}$ By definition of f:

$$f(X) \cap f(Y) = X \cap \widehat{\mathcal{C}(A)} \cap Y \cap \widehat{\mathcal{C}(B)}$$

and

$$f(X \cup Y) = (X \cup Y) \cap \mathrm{Mod}(\mathcal{C}(\overline{X \cup Y})).$$

Since $X \cap Y \subseteq X \cup Y$, to prove Expansion, i.e., $f(X) \cap f(Y) \subseteq f(X \cup Y)$, it is enough to prove that

$$\widehat{\mathcal{C}(A)} \cap \widehat{\mathcal{C}(B)} \subseteq \mathrm{Mod}(\mathcal{C}(\overline{X \cup Y})).$$

But

$$\widehat{\mathcal{C}(A)} \cap \widehat{\mathcal{C}(B)} = \mathrm{Mod}(\mathcal{C}(A) \cup \mathcal{C}(B))$$

and $\overline{X \cup Y} = A \cap B$. It is therefore enough to show that

$$\mathrm{Mod}(\mathcal{C}(A) \cup \mathcal{C}(B)) \subseteq \mathrm{Mod}(\mathcal{C}(A \cap B)).$$

By (E") it is enough to show that:

$$\mathrm{Mod}(\mathcal{C}(\overline{\widehat{A}} \cap \overline{\widehat{B}})) \subseteq \mathrm{Mod}(\mathcal{C}(A \cap B)).$$

But $A = \overline{X} = \overline{\widehat{A}}$ and $B = \overline{Y} = \overline{\widehat{B}}$.
We have proved the following.

Theorem 3 *If $\mathcal{C} : 2^{\mathcal{L}} \longrightarrow 2^{\mathcal{L}}$ satisfies Inclusion, Idempotence, Cautious Monotonicity, Conditional Monotonicity, Threshold Monotonicity and property (E), then, there is a set $\mathcal{M}$, a satisfaction relation $\models$ and a definability-preserving choice function f that satisfies Contraction, Coherence, Local Monotonicity and Expansion such that $\mathcal{C}(A) = \overline{f(\widehat{A})}$.*

Let us now turn to the soundness direction. We assume that the union of two definable set is definable, i.e.,

$$\widehat{A} \cup \widehat{B} = \mathrm{Mod}(\overline{\widehat{A} \cup \widehat{B}}) = \mathrm{Mod}(\overline{\widehat{A}} \cap \overline{\widehat{B}}). \tag{6}$$

This assumption is satisfied, for example, if the language $\mathcal{L}$ is closed under a binary connective $\vee$ that behaves semantically as a disjunction:

$$x \models a \vee b \text{ iff either } x \models a \text{ or } x \models b. \tag{7}$$

In this case, if $X = \widehat{A}$ and $Y = \widehat{B}$, then $X \cup Y = \widehat{A \vee B}$, where $A \vee B$ is the set of all formulas $a \vee b$ for $a \in A$ and $b \in B$.
We assume that f preserves definability and satisfies Contraction, Coherence, Local Monotonicity and Expansion. We also assume

$$f(X) \subseteq f(\widehat{\overline{X}}). \tag{8}$$

$\mathcal{C}$ is defined by: $\mathcal{C}(A) = \overline{f(\widehat{A})}$. We know from previous work that $\mathcal{C}$ satisfies Inclusion, Idempotence, Cautious Monotonicity, Conditional Monotonicity and Threshold Monotonicity. Therefore $\mathcal{C}$ also satisfies Cumulativity. We also know that

(9) $$\widehat{\mathcal{C}(A)} = f(\widehat{A})$$

and that

(10) $$\overline{\widehat{A}} \subseteq \bigcap_{F \supseteq A} \mathcal{C}(F).$$

We want to show that property (E) holds:

$$(\mathbf{E}) \quad \mathcal{C}(\bigcap_{F \supseteq A \text{ or } F \supseteq B} \mathcal{C}(F)) \subseteq \mathcal{C}(\mathcal{C}(A), \mathcal{C}(B)).$$

We shall proceed in two stages. In the first stage we shall show that

$$\mathcal{C}(\bigcap_{F \supseteq A \text{ or } F \supseteq B} \mathcal{C}(F)) \subseteq \mathcal{C}(\overline{\widehat{A}} \cap \overline{\widehat{B}}).$$

In the second stage we shall show that

$$\mathcal{C}(\overline{\widehat{A}} \cap \overline{\widehat{B}}) \subseteq \mathcal{C}(\mathcal{C}(A), \mathcal{C}(B)).$$

Expansion is needed only in the second stage.
Let us deal with the first stage. In fact, we can show equality. The result relies on the following lemma, asserting what Makinson [6] called Distributivity.

Lemma 1 $\mathcal{C}(A) \cap \mathcal{C}(B) \subseteq \mathcal{C}(\overline{\widehat{A}} \cap \overline{\widehat{B}})$.

Proof: By Equation 6,

$$f(\mathrm{Mod}(\overline{\widehat{A} \cup \widehat{B}})) = f(\widehat{A} \cup \widehat{B}),$$

but, by Coherence

$$f(\widehat{A} \cup \widehat{B}) \subseteq f(\widehat{A}) \cup f(\widehat{B}).$$

We conclude that we have:

$$f(\mathrm{Mod}(\overline{\widehat{A} \cup \widehat{B}})) \subseteq f(\widehat{A}) \cup f(\widehat{B}).$$

Therefore, by considering the sets of models of both sides:

$$\overline{f(\widehat{A})} \cap \overline{f(\widehat{B})} \subseteq \overline{f(\mathrm{Mod}(\overline{\widehat{A} \cup \widehat{B}}))}.$$

By the definition of $\mathcal{C}$ we have

$$\mathcal{C}(A) \cap \mathcal{C}(B) \subseteq \mathcal{C}(\overline{\widehat{A} \cup \widehat{B}}) = \mathcal{C}(\overline{\widehat{A}} \cap \overline{\widehat{B}}).$$

∎

Now, we claim:

$$\overline{\widehat{A}} \cap \overline{\widehat{B}} \subseteq \bigcap_{F \supseteq A \text{ or } F \supseteq B} \mathcal{C}(F)) \subseteq \mathcal{C}(A) \cap \mathcal{C}(B) \subseteq \mathcal{C}(\overline{\widehat{A}} \cap \overline{\widehat{B}}).$$

The first inclusion follows from Equation 10, the second inclusion is obvious and the third one follows from Lemma 1. But we know that $\mathcal{C}$ is cumulative and therefore

$$\mathcal{C}(\bigcap_{F \supseteq A \text{ or } F \supseteq B} \mathcal{C}(F)) = \mathcal{C}(\overline{\widehat{A}} \cap \overline{\widehat{B}}).$$

For the second stage, notice that, by Expansion

$$f(\widehat{A}) \cap f(\widehat{B}) \subseteq f(\widehat{A} \cup \widehat{B}).$$

By Contraction

$$f(f(\widehat{A}) \cap f(\widehat{B})) \subseteq f(\widehat{A}) \cap f(\widehat{B})$$

and by Equation 8,

$$f(\widehat{A} \cup \widehat{B}) \subseteq f(\mathrm{Mod}(\overline{\widehat{A} \cup \widehat{B}})),$$

therefore we have

$$f(f(\widehat{A}) \cap f(\widehat{B})) \subseteq f(\mathrm{Mod}(\overline{\widehat{A} \cup \widehat{B}})).$$

By considering the sets of formulas defined by both sides we have:

$$\overline{f(\mathrm{Mod}(\overline{\widehat{A} \cup \widehat{B}}))} \subseteq \overline{f(f(\widehat{A}) \cap f(\widehat{B}))}.$$

But, by Equation 9,

$$f(f(\widehat{A}) \cap f(\widehat{B})) = f(\widehat{\mathcal{C}(A)} \cap \widehat{\mathcal{C}(B)}) = f(\mathrm{Mod}(\mathcal{C}(A) \cup \mathcal{C}(B))).$$

By the definition of $\mathcal{C}$ we have:

$$\mathcal{C}(\overline{\widehat{A} \cup \widehat{B}})) \subseteq \mathcal{C}(\mathcal{C}(A), \mathcal{C}(B)),$$

which concludes the second stage of our derivation of (E). ∎

We summarize this soundness result.

Theorem 4 *If $\mathcal{M}$ is a set of models and $\models \subseteq \mathcal{M} \times \mathcal{L}$ is such that the union of any two definable sets of models is definable, and if $f : \mathcal{M} \longrightarrow \mathcal{M}$ is definability-preserving and satisfies Contraction, Coherence, Local Monotonicity, Expansion and $f(X) \subseteq f(\widehat{\overline{X}})$, then the operation defined by: $\mathcal{C}(A) = \overline{f(\widehat{A})}$ satisfies Inclusion, Idempotence, Cautious Monotonicity, Conditional Monotonicity, Threshold Monotonicity and property (E).*

One would like to prove a stronger and more elegant result. Assume that the language $\mathcal{L}$ is closed under a binary connective $\vee$. One may show, as in Theorem 4 that: if $\mathcal{M}$ is a set of models and $\models \subseteq \mathcal{M} \times \mathcal{L}$ is such that $m \models a \vee b$ iff $m \models a$ or $m \models b$, and if $f : \mathcal{M} \longrightarrow \mathcal{M}$ is definability-preserving and satisfies Contraction, Coherence, Local Monotonicity, Expansion and $f(X) \subseteq f(\widehat{\overline{X}})$, then the operation defined by: $\mathcal{C}(A) = \overline{f(\widehat{A})}$ satisfies Inclusion, Idempotence, Cautious Monotonicity, Conditional Monotonicity, Threshold Monotonicity, property (E),

(11) $$\mathcal{C}(A, a) \cap \mathcal{C}(A, b) \subseteq \mathcal{C}(A, a \vee b),$$

and

(12) $$a \in \mathcal{C}(A) \Rightarrow a \vee b \in \mathcal{C}(A),\ b \in \mathcal{C}(A) \Rightarrow a \vee b \in \mathcal{C}(A).$$

One would like to strengthen Theorem 3 by claiming: if $\mathcal{C} : 2^{\mathcal{L}} \longrightarrow 2^{\mathcal{L}}$ satisfies Inclusion, Idempotence, Cautious Monotonicity, Conditional Monotonicity, Threshold Monotonicity, property (E), (11) and (12), then, there exists a set $\mathcal{M}$, a satisfaction relation $\models$ such that $m \models a \vee b$ iff $m \models a$ or $m \models b$, and a definability-preserving choice function f that satisfies Contraction, Coherence, Local Monotonicity, Expansion and $f(X) \subseteq f(\widehat{\overline{X}})$ such that $\mathcal{C}(A) = \overline{f(\widehat{A})}$.

Unfortunately this result does not seem to be within reach, because some compactness assumption seems to be needed to show that if T is a theory that does not contain a, there is a $\vee$-complete theory $T' \supseteq T$ that does not contain a, where $\vee - complete$ means: $a \vee b \in T$ implies either $a \in T$ or $b \in T$.

A more elegant equivalence result, on the model of Theorem 8.6 of [5] may be obtained if one assumes the existence of proper negation and disjunction. We need to consider the following syntactic and semantic properties.

Weak Compactness $\mathcal{C}(A) = \mathcal{L} \Rightarrow \exists$ a finite $B \subseteq_f A$ such that $\mathcal{C}(B) = \mathcal{L}$.

(13) $$\mathcal{C}(A, a, \neg a) = \mathcal{L}$$

$$\mathcal{C}(A, \neg a) = \mathcal{L} \Rightarrow a \in \mathcal{C}(A) \tag{14}$$

$$x \models \neg a \text{ iff } x \not\models a. \tag{15}$$

Theorem 5 *Assume $\mathcal{L}$ is closed under a unary connective ($\neg$) and a binary connective ($\vee$) and that $\mathcal{C} : 2^{\mathcal{L}} \longrightarrow 2^{\mathcal{L}}$ satisfies* **Weak Compactness***, (13), (14),(11), (12), Inclusion, Idempotence, Cautious Monotonicity, Conditional Monotonicity and Threshold Monotonicity. The two following conditions are equivalent:*

1. *$\mathcal{C}$ satisfies property (E), and*
2. *there is a set $\mathcal{M}$, a satisfaction relation $\models$ that satisfies (15), (7) and a definability-preserving choice function f that satisfies Contraction, Coherence, Local Monotonicity, Expansion and (8) such that $\mathcal{C}(A) = \overline{f(\widehat{A})}$, for any $A \subseteq \mathcal{L}$.*

Proof: Assume 1. By Theorem 8.6 of [5] and Theorem 3, 2 follows. Assume 2. The union of two definable sets of models is definable, as explained following (7). Theorem 4 implies that property (E) holds. ∎

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