Nonmonotonic Logics and Semantics *

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Abstract

Tarski gave a general semantics for deductive reasoning: a formula a may be deduced from a set A of formulas iff a holds in all models in which each of the elements of A holds. A more liberal semantics has been considered: a formula a may be deduced from a set A of formulas iff a holds in all of the *preferred* models in which all the elements of A hold. Shoham proposed that the notion of *preferred* models be defined by a partial ordering on the models of the underlying language. A more general semantics is described in this paper, based on a set of natural properties of choice functions. This semantics is here shown to be equivalent to a semantics based on comparing the relative *importance* of sets of models, by what amounts to a qualitative probability measure. The consequence operations defined by the equivalent semantics are then characterized by a weakening of Tarski's properties in which the monotonicity requirement is replaced by three weaker conditions. Classical propositional connectives are characterized by natural introduction-elimination rules in a nonmonotonic setting. Even in the nonmonotonic setting, one obtains classical propositional logic, thus showing that monotonicity is not required to justify classical propositional connectives.

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1 Introduction

This paper is intended for logicians. It builds on the insights, motivations and techniques developed by researchers in Knowledge Representation and Artificial Intelligence, but its purpose is to present the topic of (AI-type) nonmonotonic deduction (or induction) to logicians. It is not claimed that the results of this paper will prove useful to AI practice. It uses the language of Mathematics (theorems and proofs) to study a form of deduction that is more general than the one used in Mathematics. A logician interested only in the (monotonic) kind of deduction used in Mathematics should not read further.

A semantics for nonmonotonic reasoning, more general than Shoham's [37], will be presented. This semantics is closely related to, but generalizes, concepts developed by the Social Choice community. In this semantic framework one may define the family of preferential operations of [19] in a way that does not assume a pre-existing monotonic logic or connectives. Connectives may then be defined and studied by introduction-elimination rules as is done in monotonic logics.

2 Monotonic Logics

In the thirties, Tarski made a number of fundamental advances in the study of mathematical logic: he proposed a semantics for logical deduction (see in particular [40, p. 127]): a formula a follows from a set A of formulas iff a holds in all models in which all the elements of A hold. These ideas were first expounded in [39, 38] (English translation in [41], Chapters 3 and 5 respectively). He characterized the consequence operations that may be defined by such a semantics as those operations that satisfy Inclusion, Idempotence and Monotonicity, as in Theorem 1. That theorem, however, does not seem to appear in Tarski's work, since he deals from the start only with compact operations. After having settled the question of what is deduction, or what is a logic, without being tied to any specific logical calculus, he was able to deal with the meaning of connectives, one at a time. This section contains a sketch of some results concerning monotonic deductive operations, that we are interested in generalizing to nonmonotonic operations.

Let us assume a non-empty set (language) $\mathcal{L}$ and a function $\mathcal{C} : 2^{\mathcal{L}} \longrightarrow 2^{\mathcal{L}}$ are given. Nothing is assumed about the language. Assume $\mathcal{M}$ is a set (of models), about which no assumption is made, and $\models \subseteq \mathcal{M} \times \mathcal{L}$ is a (satisfaction) binary relation. For any set $A \subseteq \mathcal{L}$, we shall denote by $\widehat{A}$ or by $\mathrm{Mod}(A)$ the set of all models that satisfy all elements of A:

$$\widehat{A} = \mathrm{Mod}(A) = \{x \in \mathcal{M} \mid x \models a, \, \forall a \in A\}.$$

For typographical reasons we shall use both notations, sometimes even in the same formula. For any set of models $X \subseteq \mathcal{M}$, we shall denote by $\overline{X}$ the set of all formulas that are satisfied in all elements of X:

$$\overline{X} = \{a \in \mathcal{L} \mid x \models a, \forall x \in X\}.$$

The following are easily proven, for any $A, B \subseteq \mathcal{L}$, $X, Y \subseteq \mathcal{M}$: they amount to the fact that the operations $X \mapsto \overline{X}$ and $A \mapsto \widehat{A}$ form a Galois connection.

$$A \subseteq \overline{\widehat{A}} \quad , \quad X \subseteq \widehat{\overline{X}}$$

$$\begin{array}{rcl} \widehat{A \cup B} = \widehat{A} \cap \widehat{B} & , & \overline{X \cup Y} = \overline{X} \cap \overline{Y} \\ A \subseteq B \Rightarrow \widehat{B} \subseteq \widehat{A} & , & X \subseteq Y \Rightarrow \overline{Y} \subseteq \overline{X} \\ A \subseteq B \Rightarrow \overline{\widehat{A}} \subseteq \overline{\widehat{B}} & , & X \subseteq Y \Rightarrow \widehat{\overline{X}} \subseteq \widehat{\overline{Y}} \\ \widehat{A} = \widehat{\overline{\widehat{A}}} & , & \overline{X} = \overline{\widehat{\overline{X}}} \end{array}$$

Theorem 1 *There exists a set $\mathcal{M}$ (of models) and a satisfaction relation $\models \subseteq \mathcal{M} \times \mathcal{L}$ such that $\mathcal{C}(A) = \overline{\widehat{A}}$ iff $\mathcal{C}$ satisfies the following three conditions:*

$$\textbf{Inclusion} \quad A \subseteq \mathcal{C}(A),$$

$$\textbf{Idempotence} \quad \mathcal{C}(\mathcal{C}(A)) = \mathcal{C}(A),$$

$$\textbf{Monotonicity} \quad A \subseteq B \Rightarrow \mathcal{C}(A) \subseteq \mathcal{C}(B).$$

Proof: The *only if* part is very easy to prove, using the Galois connection properties of the transformations $X \mapsto \overline{X}$ and $A \mapsto \widehat{A}$.

The *if* part takes $\mathcal{M}$ to be the set of all *theories*, the set of all sets $T \subseteq \mathcal{L}$ such that $T = \mathcal{C}(T)$. Define, then, $T \models a$ iff $a \in T$. By the definition of $\models$, $\overline{X} = \cap_{T \in X} T$ and $\widehat{A}$ is the set of theories T that include A. $\overline{\widehat{A}}$ is therefore the intersection of all theories that include A. Since, by Idempotence, $\mathcal{C}(A)$ is a theory, and since it includes A by Inclusion, $\overline{\widehat{A}} \subseteq \mathcal{C}(A)$. By Monotonicity

$\mathcal{C}(A) \subseteq \mathcal{C}(T) = T$ for any theory T that includes A, and therefore $\mathcal{C}(A)$ is a subset of the intersection of all such theories T, i.e., $\mathcal{C}(A) \subseteq \overline{\widehat{A}}$. ∎

As customary in the literature, $\mathcal{C}(A, B)$ will denote $\mathcal{C}(A \cup B)$, $\mathcal{C}(a)$ denotes $\mathcal{C}(\{a\})$ and $\mathcal{C}(A, a)$ denotes $\mathcal{C}(A \cup \{a\})$. A number of important results about the operations that satisfy Inclusion, Idempotence and Monotonicity have been proven. Let us mention three of them, in order to consider their generalization to nonmonotonic operations. The first one is that an intersection of theories is a theory. If, for any $i \in I$, $A_i = \mathcal{C}(A_i)$, then $\bigcap_{i \in I} A_i = \mathcal{C}(\bigcap_{i \in I} A_i)$. The second one is that given a family $\mathcal{C}_i$, $i \in I$, of operations that satisfy Inclusion, Idempotence and Monotonicity, their intersection, defined by, $(\bigcap_{i \in I} \mathcal{C})(A) \stackrel{\text{def}}{=} \bigcap_{i \in I}(\mathcal{C}(A))$ also satisfies Inclusion, Idempotence and Monotonicity. Another result, worth noticing, is that, if $\mathcal{C}$ satisfies Inclusion, Idempotence and Monotonicity, and if $B \subseteq \mathcal{L}$, then the operation $\mathcal{C}'$ defined by $\mathcal{C}'(A) = \mathcal{C}(A, B)$, i.e., the operation that follows from the acceptance of B once and for all satisfies Inclusion, Idempotence and Monotonicity. A last, for us, important result of Tarski that will be generalized in Section 8.2 is that the propositional connectives may be characterized elegantly and one at a time by Introduction-Elimination rules.

3 Plan of this work

This paper proposes a new family of operations, introduced in Section 4. This family is defined by five properties, two of them introduced here for the first time. These properties replace Tarski's condition of Monotonicity by three weaker properties. The main purpose of this paper is to show the importance of this family, and why it crops up naturally in different contexts. To this effect, we, first, in Section 5, describe two very different semantics, or *ontologies*, for nonmonotonic reasoning. The equivalence of these semantics is proved, under a simplifying assumption. This equivalence lends weight to the claim that both semantics are natural and important. The first one is based on choice functions, enjoying properties that have been studied by researchers in Social Choice (Social Preferences, Rational Choice or Revealed Preference Theory may have been names more to the point) [5, 3, 36, 2] already almost half a century ago. The link between their preoccupations and ours certainly needs further study. The second one is based on a qualitative notion of a measure. Its origins may be traced to Ben-David and Ben-Eliyahu's [4]. The formalism of Friedman and Halpern's [14, 15] is the one used here.

Then, in Section 6, the simplifying logical assumption is removed and the properties we expect of choice functions in the more general framework are discussed. One of the central ideas necessary to deal with the general case *definability preservation* has been put in evidence by Schlechta's [31]. The semantics proposed are a natural generalization of Tarski's semantics. The interest and importance of the family of operations described in Section 4 are put in evidence in Section 7, where it is shown that they characterize exactly the operations defined by the semantics of Section 6. In Section 8 we discuss further properties of the operations of the family defined in Section 4 and, in particular, the characterization of propositional connectives by Introduction-Elimination rules. It is shown that the logics of the (semantically) classical connectives is classical propositional calculus. In Section 9 it is shown that, if the language $\mathcal{L}$ is a propositional calculus, the finitary consequence relations defined by our family of operations are exactly the preferential consequence relations of [19]. In Section 10 a more restricted family of operations, defined by equivalent semantic restrictions concerning choice functions on one hand and qualitative measures on the other hand, is characterized by an additional requirement on the nonmonotonic operations. Properties of this family are briefly discussed. Section 11 is a conclusion.

4 Nonmonotonic Deduction Operations

In this section we shall present five properties of an operation $\mathcal{C} : 2^{\mathcal{L}} \longrightarrow 2^{\mathcal{L}}$, discuss them and their relation to properties of monotonic operations. We shall argue that they should be satisfied by inference operations. In the next sections, we shall define suitable semantics for them and prove a representation theorem.

Our first two properties are uncontroversial, at least for mathematical logicians. They may not be satisfied by most deductive agents with bounded resources, though. Inclusion and Idempotence are two of the conditions characterizing monotonic deduction and seem most natural also in the context of nonmonotonic deduction.

$$\textbf{Inclusion} \quad \forall A \subseteq \mathcal{L} \; A \subseteq \mathcal{C}(A),$$

$$\textbf{Idempotence} \quad \forall A \subseteq \mathcal{L} \; \mathcal{C}(\mathcal{C}(A)) = \mathcal{C}(A).$$

The first one, Inclusion, requires that all assumptions be part of the conclusions. The second one, Idempotence, expresses the requirement that the strength or the validity of a conclusion be unaffected by the length of the chain of arguments leading to its acceptance. We require the operation $\mathcal{C}$ squeeze the *fruit*, i.e., the assumptions to the end, i.e., until no more conclusions can be obtained.

The next three properties are properties of restricted monotonicity. They are implied by Monotonicity, and express that, in certain situations, Monotonicity is required.

Cautious Monotonicity $\forall A, B \subseteq \mathcal{L}\ A \subseteq B \subseteq \mathcal{C}(A) \Rightarrow \mathcal{C}(A) \subseteq \mathcal{C}(B)$

Cautious Monotonicity is a restricted form of Monotonicity: any monotonic operation is cautiously monotonic. This property was first introduced by D. Gabbay [16], in its finitary form and by D. Makinson [24, 25] in its infinitary form. It requires that one does not retract previous conclusions when one learns that a previous conclusion is indeed true. It seems to have been accepted as reasonable by all researchers in the field. A discussion of its appeal may be found in [19].

The next two properties are described here for the first time, but they are closely related to the property previously discussed under the name of Deductivity or Infinite Conditionalization in [11, 12, 9, 13, 25, 18]. The first one, termed Conditional Monotonicity, expresses the requirement that $\mathcal{C}$ behave monotonically if one replaces, in the assumptions, some of the assumptions by their consequences. It asserts that non-monotonicity cannot be caused by the deduction process itself. It is only the addition of new assumptions unrelated to the old ones or assumptions that are less than the full sets of conclusions that can lead to non-monotonicity.

Conditional Monotonicity $\forall A, B \subseteq \mathcal{L},\ \mathcal{C}(A, B) \subseteq \mathcal{C}(\mathcal{C}(A), B)$

This is indeed a monotonicity requirement, if $\mathcal{C}$ satisfies Inclusion, since $A \cup B \subseteq \mathcal{C}(A) \cup B$. But this is a very restricted form of Monotonicity: we do not allow adding arbitrary, unrelated, assumptions, only replacing part of the assumptions by their consequences. When doing so, we do require that one does not lose conclusions, but one may add conclusions. It is worth noticing that Conditional Monotonicity is an intrinsically infinitary condition: even if A and B are finite, $\mathcal{C}(A)$ that appears on the right hand side is typically infinite. If $\mathcal{C}$ is a Tarski deductive operation, i.e., satisfies Inclusion, Idempotence and Monotonicity, then,

$$\mathcal{C}(A,B) \subseteq \mathcal{C}(\mathcal{C}(A),B) \subseteq \mathcal{C}(\mathcal{C}(A),\mathcal{C}(B)) \subseteq \mathcal{C}(\mathcal{C}(A,B)) \subseteq \mathcal{C}(A,B)$$

and we have: $\mathcal{C}(A,B) = \mathcal{C}(\mathcal{C}(A),B)$. In the system presented here, $\mathcal{C}(A,B)$ may be a strict subset of $\mathcal{C}(\mathcal{C}(A),B)$. Conditional Monotonicity seems related with Cautious Monotonicity, but it is not. In Cautious Monotonicity, we allow the addition of a subset of $\mathcal{C}(A)$, whereas Conditional Monotonicity requires the addition of $\mathcal{C}(A)$ in its entirety. In Conditional Monotonicity, the addition may be done in the presence of some other assumptions (B), whereas in Cautious Monotonicity B must be empty. The intuitive justification for Conditional Monotonicity will be given now, it explains the term Conditional Monotonicity: if $c \in \mathcal{C}(A,B)$, then, in $\mathcal{C}(A)$, there should be something (some conditional statements) to the effect that: *if* B *then* c. But, together with B, this should imply c and therefore c should be in $\mathcal{C}(\mathcal{C}(A),B)$.

Our last property, termed Threshold Monotonicity, requires that $\mathcal{C}$ behave monotonically in all cases in which the deduction process has already been applied to part of the assumptions, i.e., above the threshold of some $\mathcal{C}(A)$. If $\mathcal{C}(A)$ is part of the assumptions, then $\mathcal{C}$ behaves monotonically.

Threshold Monotonicity $\quad \mathcal{C}(A) \subseteq B \subseteq C \ \Rightarrow\ \mathcal{C}(B) \subseteq \mathcal{C}(C)$

The intuitive reason for this requirement is similar to the one for Conditional Monotonicity. We expect that, if $\mathcal{C}(A) \subseteq B$ and $c \in \mathcal{C}(B)$, formulas saying that: *if* B *then* c, are already in $\mathcal{C}(A)$. The set C, then, contains both *if* B *then* c and B, therefore $\mathcal{C}(C)$ should contain c.

Before we move to our main representation result, let us draw one important consequence of the properties above.

Lemma 1 (Cumulativity, Makinson [25]) *If* $\mathcal{C}$ *satisfies Idempotence and Cautious Monotonicity, then it satisfies*

Cumulativity $\quad A \subseteq B \subseteq \mathcal{C}(A) \Rightarrow \mathcal{C}(B) = \mathcal{C}(A)$.

The importance of Cumulativity has been stressed early on by Makinson [24].
Proof: By Cautious Monotonicity, $\mathcal{C}(A) \subseteq \mathcal{C}(B)$. Therefore we have $B \subseteq \mathcal{C}(A) \subseteq \mathcal{C}(B)$. By Cautious Monotonicity again, we have $\mathcal{C}(B) \subseteq \mathcal{C}(\mathcal{C}(A))$. By Idempotence, then $\mathcal{C}(B) \subseteq \mathcal{C}(A)$. ∎

5 Two semantics in a simplified framework

We shall now describe two natural different semantics for nonmonotonic operations: one based on choice functions and one based on qualitative probability measures. We shall show their equivalence, under simplifying assumptions to be described now. The exact fit between the semantics based on choice functions and the formal properties of nonmonotonic deduction operations described in Section 4 will be proved in Section 7.

5.1 A simplifying assumption

The general setting, assuming an arbitrary language $\mathcal{L}$, will be developed in Section 6. It requires that sets of models that may be defined by a set of formulas be given prominence.

Definition 1 *A set X of models is said to be definable iff one of the two following equivalent conditions holds:*

1. *$\exists A \subseteq \mathcal{L}$ such that $X = \widehat{A}$, or*
2. *$X = \widehat{\overline{X}}$.*

The set of all definable subsets of X will be denoted by D_X.

A set of models is definable iff it is the set of all models satisfying some set of formulas. In many situations in which a *finiteness* assumption is reasonable, one may avoid the consideration of the special role of definable sets. For example, researchers in Social Choice typically assume the set of outcomes is finite. Friedman and Halpern, also, assume, at least in part of their work, that the base set is finite. For expository purposes, to keep definitions and justifications simple, we shall now make a similar assumption, to be lifted in Section 6. We shall assume that every set of models is definable.

Simplifying Assumption $\forall X \subseteq \mathcal{M},\ X = \widehat{\overline{X}}$

Any propositional calculus on a *finite* number of atomic propositions satisfies our Simplifying Assumption.

5.2 Choice functions

The properties described in this section have been put in evidence for the first time, probably, by researchers in social choice, triggered by H. Chernoff [5], H. Uzawa [42] and K. Arrow [3] (for an updated survey, see [27]). The exact nature of the link between nonmonotonic logics and the theory of choice functions needs further research. For the sake of those readers who are not familiar with this literature, let us describe briefly its framework: each individual has personal preferences over a set of possible outcomes. The society, given a subset of those, the feasible outcomes, must come up with a the subset of those feasible outcomes that are acceptable socially, in view of the individual preferences. Different methods of social decision result in different functions from sets of feasible outcomes to sets of acceptable outcomes. Social Choice investigates the relations between those different methods for social decision and the choice functions they determine.

Independently, Y. Shoham, in [37], proposed a general semantics for nonmonotonic reasoning, based on preferences among models. The link between the properties of choice functions studied by Social Choice researchers and Nonmonotonic Reasoning has been put in evidence by Doyle and Wellman [7], Rott [30] and Lindström [23]. Lindström generalizes the finitary framework considered in Revealed Preference Theory to an infinitary framework. In [31, 32], Karl Schlechta rediscovered choice functions and their properties, in the infinitary framework. He also considered an additional property needed in such a framework. Schlechta's line of research is best described in [34] and [35]. The only novelty of this section is the detailed argumentation justifying the assumptions about the choice function.

The basic idea is that one can generalize Tarski's semantic analysis of deduction by considering, instead of all models of a set A of formulas, only a subset of this set: the set of *best* models of A, jumping, on the basis of A, to the conclusion that the situation at hand is one of those *best* situations. Such a manner of drawing conclusions from categorical information has been time and again attested by researchers in cognitive sciences. Lakoff's [20] is a good introduction. The author's [21] represents a very tentative formalization.

We consider a set $\mathcal{M}$ of models, a satisfaction relation $\models$ and a choice function $f : 2^{\mathcal{M}} \longrightarrow 2^{\mathcal{M}}$ that chooses, for a set X of models, the set of *best*, *most typical*, *most important*, or *preferred* models for the set X. Then we define the function $\mathcal{C}$ by

$$\mathcal{C}(A) = \overline{f(\widehat{A})}. \tag{1}$$

The use of a choice function to define a (nonmonotonic) consequence operation generalizes Shoham's semantics and represents one step up in the abstraction ladder from Shoham's semantics. Shoham assumed that the choice function f is defined in a particular way: $f(X)$ is the set of all elements of X that are *minimal* in X, under some, pre-existing, order relation on $\mathcal{M}$. We prefer to deal directly with properties of the function f, without assuming any order relation on $\mathcal{M}$.

Let us now present natural conditions on f. The monotonic case corresponds to the case that the function f is required to be the identity function. The conditions below are trivially satisfied by the identity function.

Let us take the following running example, that will exemplify the fact that the properties of the choice function f define, in a sense the logic of optimization. You are looking for an apartment in Paris. The set of apartments on the market in Paris is the set $\mathcal{M}$. Your real estate agent asks for your desiderata and your financial possibilities. You expect her to come up with a limited list of the *best* apartments available in Paris. You expect a list that is neither too small nor too large. Notice that this example does not fit exactly the Social Choice paradigm, interested in finding a subset of outcomes that are acceptable in the view of contradictory preferences of the individuals, since we assumed you alone are looking for an apartment. But it may be fitted to the Social Choice paradigm if you consider that you are shopping for an apartment that fits in some way the contradictory desiderata of a family, and of the real estate agent.

The first property we expect of f is a property of contraction.

Contraction $\quad f(X) \subseteq X$

Indeed, intuitively, f picks the *preferred* models of the set X and those models are in the set X. The identity function satisfies Contraction. Contraction is assumed in the Social Choice literature without even mentioning it explicitly.

If you requested an apartment in Paris, you expect to get a list of apartments in Paris, not in Neuilly or in San Francisco.

The second property expresses the fact that, if X is a subset of Y, it is more difficult to be one of the best of the bigger set Y than to be one of the best of the smaller set X. Therefore we expect any element of $f(Y)$ that happens to be in X to be in $f(X)$.

Coherence $\quad X \subseteq Y \Rightarrow X \cap f(Y) \subseteq f(X)$

This is Sen's [36] property α. The identity function satisfies Coherence. Coherence is a kind of antimonotonicity: if $X \subseteq Y$, then antimonotonicity would require: $f(Y) \subseteq f(X)$, whereas Coherence only requires that this part of $f(Y)$ that is included in X is included in $f(X)$. The term Coherence seems to be an appropriate name since it expresses the existence of some kind of coherent test by which the *preferred* elements of a set are picked up: the test corresponding to a superset must be at least as demanding as the one of a subset. The Coherence property appears in Chernoff's [5] and has been given his name in [27]. It has been endorsed by all researchers in Social Choice.

To illustrate Coherence, suppose you now remember you promised your wife you would live on the left bank, but forgot to tell that to your agent. You tell her that and get a new list, of apartments on the left bank. You expect all the left bank apartments that appeared in the first list to be included in the second list. The second list will probably also include other apartments, that were not part of the, say twenty, best apartments in Paris, but are part the best apartments on the left bank.

The third and last property expresses the fact that, if Y is a subset of X, but large enough to include all of $f(X)$, then we do not expect $f(Y)$ to be larger than $f(X)$.

Local Monotonicity $\quad f(X) \subseteq Y \subseteq X \Rightarrow f(Y) \subseteq f(X)$

The term Local Monotonicity expresses that this is a property of qualified monotonicity for f: $Y \subseteq X$ implies $f(Y) \subseteq f(X)$, conditional on $f(X) \subseteq Y$, somehow a local condition. The importance of property has been put in evidence by M. A. Aizerman [2, 1]. The identity function satisfies Local Monotonicity, because it is monotonic, but also trivially, because the assumption implies $X = Y$.

Suppose all best apartments of Paris included in the list you got from your agent happen to be on the left bank. You certainly would not expect a larger list if you told her you want only apartments on the left bank.

Any choice function satisfying Contraction, Coherence and Local Monotonicity is considered acceptable in this work. A number of other properties of choice functions, from the literature, will be discussed now. Researchers in Social Choice have universally endorsed the following: $f(X)$ is not empty if X is not empty. Some have noticed that this property is not crucial and that they could build the theory without it. We have no reason to make this requirement: it may well be the case that our search for a large cheap apartment in Paris in the best quarter leaves us with an empty list.

Another property that has been widely considered is:

Expansion $f(X) \cap f(Y) \subseteq f(X \cup Y)$

It does not follow from Contraction, Coherence and Local Monotonicity. We do not endorse it. An apartment, on the Ile de la Cité, that makes the list of the ten best apartments on the left bank and (or) the Ile de la Cité and also makes the list of the ten best apartments on the right bank and (or) the Ile de la Cité, does not necessarily makes the list of the ten best apartments in Paris. One easily sees that the semantics proposed by Shoham in [37] validates Contraction, Coherence, Local Monotonicity and also Expansion: if z is minimal in X and minimal in Y it is minimal in the union $X \cup Y$. Therefore, the semantics proposed in this paper is a strict generalization of Shoham's. In fact, under our simplifying assumption, Shoham's semantics is equivalent to considering choice functions that satisfy Contraction, Coherence, Local Monotonicity and Expansion.

We have presented a semantic framework that generalizes Tarski's. It involves a choice function on sets of models. This choice function is assumed to satisfy three conditions. We argued that those three conditions are natural, but the ultimate test of their interest lies ahead. The Social Choice literature mentions two main results about choice functions that satisfy Contraction, Coherence and Local Monotonicity. In [28], it is shown that they are exactly the choice functions that satisfy Contraction and

Path Independence $f(X \cup Y) = f(f(X) \cup Y)$.

In [2], it is shown that, if $\mathcal{M}$ is finite, they are exactly the *pseudo-rationalizable* choice functions, i.e., those that may be defined by a finite set of binary *preference* relations $>_i$ on $\mathcal{M}$ by taking, for $f(X)$, the set of all elements of X that are minimal in X for at least one of the $>_i$'s. None of these results will be used in this paper.

5.3 Qualitative Measures

A completely different generalization of Tarski's semantic analysis will be reviewed now. Its origins may be traced to Dubois and Prade [8] and Ben-David and Ben-Eliyahu [4]. Up to small technical changes, our presentation will be that of Friedman and Halpern [14, 15]. The connection between both approaches is described in [33]. Some more results concerning the link between plausibility measures and preferential relations may be found in [10].

Suppose we had some way of measuring the *size* or the *importance* of sets of models. One is tempted to say that a formula a may be deduced from a set A of formulas iff the measure of the set of all models of A that satisfy a is larger than that of the set of models of A that do not satisfy a. Since, as in the case of monotonic logic, one would like to deduce anything from an inconsistent set of formulas, the case that the set of models of A is negligible, i.e., not larger than the empty set, has to be treated separately. With the notations of Section 5.2, one would like to assume a binary relation $>$ on $2^{\mathcal{M}}$ ($X > Y$ iff the measure of X is larger than that of Y) and define the deductive operation by:

(2) $$a \in \mathcal{C}(A) \text{ iff either } \widehat{A} \cap \widehat{\{a\}} > \widehat{A} - \widehat{\{a\}} \text{ or } \widehat{A} \not> \emptyset.$$

This definition indeed generalizes Tarski's. For Tarski: $X > Y$ iff $Y = \emptyset$ and $X \neq \emptyset$.

The following list of natural properties for $>$ is similar to Friedman and Halpern's definition of a Qualitative Plausibility Measure.[1] The properties described here are in fact stronger than theirs: the representation result holds for both sets of properties. First, it seems reasonable to require that $>$ be a strict partial order relation, i.e., irreflexive and transitive. This seems implied by our description of $X > Y$ as meaning that the *measure* of X is larger than that of Y. Note that we do not require the relation $>$ to be total, which would not be reasonable since X and Y may have equal measure without being equal, nor even to be *modular*, i.e., satisfy the property $X > Y$ implies that, for any Z, either $X > Z$ or $Z > Y$. Modularity will be assumed in Section 10.

(3) The relation $>$ is irreflexive and transitive.

[1] Friedman and Halpern assume a function $Pl : 2^{\mathcal{M}} \mapsto D$ and a reflexive, transitive and anti-symmetric relation $\leq$ on D satisfying:

- **A1** if $X \subseteq Y$ then $Pl(X) \leq Pl(Y)$,
- **A2** if X, Y, Z are pairwise disjoint sets, $Pl(X \cup Y) \geq Pl(Z)$ and $Pl(X \cup Z) \geq Pl(Y)$, then $Pl(X) \geq Pl(Y \cup Z)$,
- **A3** if $Pl(X) = Pl(Y) = Pl(\emptyset)$, then $Pl(X \cup Y) = Pl(\emptyset)$.

A second very natural property of $>$ is that it should behave as expected with respect to set inclusion. The reader will easily be convinced that, in our setting, the correct formulation is the following:

$$W \supseteq X > Y \supseteq Z \;\Rightarrow\; W > Z. \tag{4}$$

Our third property deals with the special character of the empty set. The empty set is the ultimate *small* set and any set X that is not strictly greater than the empty set must be extremely small and negligible. In many cases, any nonempty set will be strictly greater than the empty set, but we do not wish to make this a requirement. Since the union of any family of empty sets is empty, it is reasonable to require that the union of a family of negligible sets be negligible.

$$\forall i \in I, X_i \not> \emptyset \;\Rightarrow\; \bigcup_{i \in I} X_i \not> \emptyset \tag{5}$$

This is the infinitary version of property (A3) of Friedman and Halpern. The finitary version is not enough for Theorem 3 to hold. The next two properties have to deal with the qualitative character of the relation $>$. Qualitatively greater has to be understood here as *an order of magnitude* greater. Assume $X \cup Y > Y$. If $X \cup Y$ is an order of magnitude greater than Y, it must be that X is already greater than Y.

$$X \cup Y > Y \;\Rightarrow\; X > Y \tag{6}$$

Note that the definition of $\mathcal{C}$ in 2 makes use of the relation $>$ only between sets with an empty intersection. Property 6 can therefore only have an indirect influence. The qualitative plausibility measures of Friedman and Halpern need not satisfy Property 6. The results presented in Section 5.4 show that one may add this property without harm. Friedman and Halpern consider a property (A2) that implies the finitary version of 7. The finitary version of 7 together with 6 imply (A2). The next and last property is the fundamental one that makes the qualitative character of $>$ apparent.

$$\forall i \in I,\; X > Y_i \;\Rightarrow\; X > \bigcup_{i \in I} Y_i \tag{7}$$

A set that is greater than every one of a family of sets must be greater than their union: pooling small sets never makes a big set. This seems to be the essence of *qualitative*.

We have presented a set of properties for $>$ and have argued that they are natural properties for a qualitative measure. No such argument can be completely convincing. In the next section, it will be shown that the properties 3, 4, 5, 6 and 7 for $>$ are equivalent to the conditions of Contraction, Coherence and Local Monotonicity for f. This equivalence suggests that those two sets of properties and the nonmonotonic operations they define have a central role to play in the study of nonmonotonic logics. To the best of my knowledge, the qualitative measures have not been studied by the Social Choice community. Since, as will be seen in Section 5.4, they are equivalent to choice functions, it may be worthwhile to ask whether they can help there. The property of Expansion considered in Social Choice translates readily into $\bigcup_{i \in I} X_i > Z$ implies that there is some $i \in I$ such that $X_i > Z$.

5.4 A Semantic Equivalence

Theorem 2 *Suppose f is a choice function that satisfies Contraction, Coherence and Local Monotonicity. Then, the relation $>$ defined by: $X > Y$ iff $f(X) \neq \emptyset$ and $Y \cap f(X \cup Y) = \emptyset$ satisfies properties 3, 4, 5, 6 and 7. If $\mathcal{C}$ is defined by Equation 1, then it satisfies Equation 2.*

Before we get to the proof, let us ponder on the translation proposed. Is it a natural translation, i.e., does it fit the intuitive interpretations given to $>$ and f? The sets X such that $f(X) = \emptyset$ are the *no solution* sets (in our running examples: no suitable apartments in X). They may be assumed to be small. If X is qualitatively larger than Y, it is reasonable to assume that X is not so small as to have an empty image under f. In Section 5.3, the intuition we developed was that X is qualitatively larger than Y means that X contains some important elements Y does not contain. The important elements are those of $f(X)$ and Y contains none of them.

Proof: Let us show, first, that $\mathcal{C}$ satisfies Equation 2. Obviously $a \in \overline{f(\widehat{A})}$ iff $f(\widehat{A}) \subseteq \widehat{\{a\}}$. By Contraction, we have

$$f(\widehat{A}) \subseteq \widehat{\{a\}} \text{ iff } (\widehat{A} - \widehat{\{a\}}) \cap f(\widehat{A}) = \emptyset.$$

The only thing left for us to check is that if $(\widehat{A} - \widehat{\{a\}}) \cap f(\widehat{A}) = \emptyset$ and $f(\widehat{A}) \neq \emptyset$, then $f(\widehat{A} \cap \widehat{\{a\}}) \neq \emptyset$, which follows from Coherence.

Irreflexivity of $>$ follows from Contraction. For Transitivity, assume $f(X) \neq \emptyset$, $Y \cap f(X \cup Y) = \emptyset$, $f(Y) \neq \emptyset$ and $Z \cap f(Y \cup Z) = \emptyset$. We want to show that $Z \cap f(X \cup Z) = \emptyset$. By Contraction, we know that $f(X \cup Y \cup Z) \subseteq (X - Z) \cup (Y \cup Z) \cap f(X \cup Y \cup Z)$. By Coherence, $(Y \cup Z) \cap f(X \cup Y \cup Z) \subseteq f(Y \cup Z)$. By assumption, $f(Y \cup Z) \subseteq Y - Z$. We conclude that $f(X \cup Y \cup Z) \subseteq (X \cup Y) - Z$ and therefore $f(X \cup Y \cup Z) \subseteq X \cup Y$. By Coherence again, then $f(X \cup Y \cup Z) \subseteq f(X \cup Y)$ and, by assumption, $f(X \cup Y \cup Z) \subseteq X$. We have $f(X \cup Y \cup Z) \subseteq X \cup Z \subseteq X \cup Y \cup Z$. By Local Monotonicity, then, we conclude that $f(X \cup Z) \subseteq f(X \cup Y \cup Z)$. Therefore $f(X \cup Z) \subseteq (X \cup Y) - Z$. We conclude that $Z \cap f(X \cup Z) = \emptyset$.

For property 4, assume $X \supseteq Y$, $W \supseteq Z$, $f(Y) \neq \emptyset$ and $W \cap f(Y \cup W) = \emptyset$. Assume $f(X) = \emptyset$. We have $f(X) \subseteq Y \subseteq X$ and, by Local Monotonicity, $f(Y) \subseteq f(X)$ and $f(X) \neq \emptyset$. We have shown that $f(X) \neq \emptyset$. We have $f(Y \cup W) \subseteq Y \subseteq Y \cup Z \subseteq Y \cup W$. By Local Monotonicity, then, we have $f(Y \cup Z) \subseteq f(Y \cup W)$. By Coherence,

$$(Y \cup Z) \cap f(X \cup Z) \subseteq f(Y \cup Z) \subseteq f(Y \cup W) \subseteq Y - W \subseteq Y - Z.$$

We conclude that $Z \cap f(X \cup Z) = \emptyset$.

For property 5, we notice that, from the definition of $>$, $X > \emptyset$ iff $f(X) \neq \emptyset$. We have to show that $f(X_i) = \emptyset$ for any $i \in I$ implies $f(\bigcup_{i \in I} X_i) = \emptyset$. This follows from Contraction and Coherence.

For property 6, assume $f(X \cup Y) \neq \emptyset$ and $Y \cap f(X \cup Y) = \emptyset$. The only thing left for us to prove is that we have $f(X) \neq \emptyset$. Notice that, by Coherence, $X \cap f(X \cup Y) \subseteq f(X)$. It is enough to show that $X \cap f(X \cup Y) \neq \emptyset$. By Contraction, $f(X \cup Y) \subseteq X \cup Y$ and, therefore,

$$f(X \cup Y) \subseteq X \cap f(X \cup Y) \cup Y \cap f(X \cup Y) = X \cap f(X \cup Y),$$

by assumption.

Before we prove the last property needed, 7, let us prove a lemma.

Lemma 2 $f(\bigcup_{i \in I} X_i) \subseteq \bigcup_{i \in I} f(X_i)$.

Proof: By Coherence, we have $X_i \cap f(\bigcup_{i \in I} X_i) \subseteq f(X_i)$. By Contraction, $f(\bigcup_{i \in I} X_i) \subseteq \bigcup_{i \in I} X_i$. ∎

For property 7, assume $f(X) \neq \emptyset$ and $Y_i \cap f(X \cup Y_i) = \emptyset$ for any $i \in I$. By Lemma 2, $f(X \cup \bigcup_{i \in I} Y_i) \subseteq \bigcup_{i \in I} f(X \cup Y_i)$. But, by assumption, $f(X \cup Y_i) \subseteq X$, for any $i \in I$. Therefore, for any $j \in I$, $f(X \cup \bigcup_{i \in I} Y_i) \subseteq X \cup Y_j$. By Coherence, then, $(X \cup Y_j) \cap f(X \cup \bigcup_{i \in I} Y_i) \subseteq f(X \cup Y_j) \subseteq X - Y_j$. We have shown that, for any $j \in I$, $f(X \cup \bigcup_{i \in I} Y_i) \subseteq X - Y_j$. We conclude that

$$\bigcup_{j \in I} Y_j \cap f(X \cup \bigcup_{i \in I} Y_i) = \emptyset.$$

$\blacksquare$

Before we get to the second leg of our equivalence trip, let us notice an additional property.

Definition 2 *If $x \in X$ we shall say that x is* heavy *in X iff $X \not> \{x\}$.*

Lemma 3 *If the relation $>$ is defined as in Theorem 2, then, if $f(X) = \emptyset$ all the members of X are heavy, and if $f(X) \neq \emptyset$, the heavy elements of X are precisely the members of $f(X)$.*

The proof is obvious.

The second half of the equivalence between choice functions and qualitative measures will be described now.

Theorem 3 *Suppose $>$ is a qualitative measure that satisfies properties 3, 4, 5, 6 and 7. Then, the choice function f defined by taking for $f(X)$ the set of heavy elements (see Definition 2) of X satisfies Contraction, Coherence and Local Monotonicity. If $\mathcal{C}$ is defined by Equation 2, it satisfies Equation 1.*

Proof: By definition, f satisfies Contraction. For Coherence, assume $X \subseteq Y$ and $x \in X$ is heavy in Y, i.e., $Y \not> \{x\}$. By property 4, $X \not> \{x\}$ and x is heavy in X. We have proved that f satisfies Coherence.

For Local Monotonicity, assume $Y \subseteq X$ and all heavy elements of X are in Y. Let y be a heavy element of Y. We know that $y \in X$. We must show that y is heavy in X. Since all heavy elements in X are in Y, any member z of $X - Y$ is not heavy in X and therefore $X > \{z\}$. By property 7, we have $X > X - Y$. If y was not heavy in X, again by 7, we would have $X > (X - Y) \cup \{y\}$. By property 6, then, $Y - \{y\} > X - Y \cup \{y\}$. By property 4, then, we would have $Y > \{y\}$, contrary to the assumption that y is heavy in Y. We have shown that f satisfies Local Monotonicity.

We must now show that $\mathcal{C}$ satisfies Equation 1. Our proof will be exactly the same as the corresponding part of the proof of Theorem 2, once we have shown that the property used as a definition in Theorem 2 holds true.

$$X > Y \text{ iff } f(X) \neq \emptyset \text{ and } Y \cap f(X \cup Y) = \emptyset \tag{8}$$

Assume $X > Y$. By property 4, $X > \emptyset$. By property 5, there is some element of $x \in X$ such that $\{x\} > \emptyset$. The element x is heavy in $\{x\}$ since $>$ is irreflexive, and therefore $x \in f(\{x\})$. By Coherence, we conclude that

$f(X) \neq \emptyset$. By property 4, if $y \in Y$, then $X > \{y\}$ and y is not heavy in X. We have shown that $Y \cap f(X) = \emptyset$.

Suppose now that $f(X) \neq \emptyset$ and $Y \cap f(X \cup Y) = \emptyset$. For any element $y \in Y$, $X \cup Y > \{y\}$. By property 7, $X \cup Y > Y$. By property 6, $X > Y$. ∎

We have shown that the two very different semantic frameworks proposed by choice functions and qualitative measures are equivalent. This is a clear indication that the notion captured is important and that the operations $\mathcal{C}$ that may be defined in those frameworks form a class of great interest. Section 4 proposes a characterization of those operations. The representation result is proved in Section 7.

6 Choice functions: a general semantics for nonmonotonic operations

It is now time to get rid of the simplifying assumption that all sets of models are definable (see Definition 1), made in Section 5.2. In the general case, when certain sets of models are not definable, we must take a second look at the properties of choice functions considered in Section 5.2. Notice, first, that in the definition of $\mathcal{C}(A)$ as $\overline{f(\widehat{A})}$, the argument of f is a definable set $(\widehat{A})$. We have no need for applying f to a set that is not definable and, therefore, we shall assume that the domain of f is the set of definable sets of models. First, we shall require that the image under f of a definable set be definable. This requirement has been introduced by Schlechta in [31]. Then, we shall understand the variables X and Y appearing in the definition of the properties of f that we considered (Contraction, Coherence and Local Monotonicity), *not* as ranging over all sets of models X and Y, but only over *definable* such sets.

On the first point, we expect to be able to describe (by a set of formulas) the sets in which we are interested, and on which we want to apply the choice function f. But, similarly, we expect the result of the application of f, the set of *preferred* elements of a set X to be definable by a set of formulas (otherwise, how could we describe it in the language at our disposal?). We shall consider choice functions that are defined only on definable sets of models and that send definable sets to definable sets, i.e., we assume:

$$\textbf{Definability Preservation} \ \ \forall A \subseteq \mathcal{L},\ f(\widehat{A}) = \widehat{\overline{f(\widehat{A})}}$$

The identity function obviously preserves definability. This property has never been considered by the Social Choice community, which seems to have been interested so far only in the case that $\mathcal{M}$ is finite.

On the second point, for example, Local Monotonicity is now understood as: for any *definable* sets X, Y, such that $f(X) \subseteq Y \subseteq X$, one has $f(Y) \subseteq f(X)$. It turns out (the proof is left to the reader) that one may extend any f defined on definable sets (and satisfying Contraction, Coherence and Local Monotonicity) to arbitrary subsets by: $f(X) = X \cap f(\widehat{\overline{X}})$. This extension satisfies Contraction and Coherence for any subsets X, Y, but it does not satisfy Local Monotonicity for arbitrary such sets. It satisfies the following: for any sets X, Y such that $f(\widehat{\overline{X}}) \subseteq Y \subseteq X$, one has $f(Y) \subseteq f(X)$.

In the next section, the family of nonmonotonic operations defined by Equation 1 from definability-preserving choice functions that satisfy Contraction, Coherence and Local Monotonicity will be described.

7 Representation result

We shall now state and then prove our main characterization result. It proves an exact correspondence between the choice function semantics of Section 6 and the properties of $\mathcal{C}$ described in Section 4. Theorems 2 and 3 show that those properties also correspond exactly with the qualitative measures of Section 5.3, at least under the simplifying assumption of Section 5.1. Lifting this simplifying assumption there would involve a careful study of which sets of models are *measurable* in the qualitative sense. This can surely be done.

Theorem 4 *Suppose we are given a language $\mathcal{L}$ and a function $\mathcal{C} : 2^{\mathcal{L}} \longrightarrow 2^{\mathcal{L}}$. Then, the following two conditions are equivalent:*

1. *$\mathcal{C}$ satisfies Inclusion, Idempotence, Cautious Monotonicity, Conditional Monotonicity and Threshold Monotonicity,*

2. *there exists a set $\mathcal{M}$ (of models), a satisfaction relation $\models \subseteq \mathcal{M} \times \mathcal{L}$ and a definability-preserving choice function $f : D_{\mathcal{M}} \longrightarrow D_{\mathcal{M}}$ satisfying Contraction, Coherence and Local Monotonicity such that $\mathcal{C}(A) = \overline{f(\widehat{A})}$.*

The notation $D_{\mathcal{M}}$ is explained in Definition 1.

Proof: First, let us show soundness: property 2 implies property 1. Assume $\mathcal{M}$ is a set and $\models$ a binary relation on $\mathcal{M} \times \mathcal{L}$. Assume f satisfies Definability Preservation, Contraction, Coherence and Local Monotonicity. Define $\mathcal{C}(A) = \overline{f(\widehat{A})}$. We shall show that $\mathcal{C}$ satisfies Inclusion, Idempotence, Cautious Monotonicity, Conditional Monotonicity and Threshold Monotonicity.

For Inclusion, notice that, by Contraction, $f(\widehat{A}) \subseteq \widehat{A}$, and therefore $\overline{\widehat{A}} \subseteq \overline{f(\widehat{A})} = \mathcal{C}(A)$. But $A \subseteq \overline{\widehat{A}}$ and we conclude that $A \subseteq \mathcal{C}(A)$. We shall now prove a lemma that makes use of the definability preservation property.

Lemma 4 $f(\widehat{A}) = \widehat{\mathcal{C}(A)}$.

Proof: By Definability Preservation, $f(\widehat{A}) = \widehat{\overline{f(\widehat{A})}}$. But $\widehat{\overline{f(\widehat{A})}} = \widehat{\mathcal{C}(A)}$. ∎

For Idempotence, notice that, by Inclusion (already proved), we have $A \subseteq \mathcal{C}(A)$ and therefore $\widehat{\mathcal{C}(A)} \subseteq \widehat{A}$. By Coherence, then, we have $\widehat{\mathcal{C}(A)} \cap f(\widehat{A}) \subseteq f(\widehat{\mathcal{C}(A)})$. By Lemma 4, $\widehat{\mathcal{C}(A)} \cap f(\widehat{A}) = f(\widehat{A})$ and $f(\widehat{A}) \subseteq f(\widehat{\mathcal{C}(A)})$. We conclude that we have $\mathcal{C}(\mathcal{C}(A)) \subseteq \mathcal{C}(A)$. The opposite inclusion follows from Inclusion.

For Cautious Monotonicity, we use Local Monotonicity and Lemma 4. Assume $A \subseteq B \subseteq \mathcal{C}(A)$. We have $\widehat{\mathcal{C}(A)} \subseteq \widehat{B} \subseteq \widehat{A}$. By Lemma 4, $f(\widehat{\overline{\widehat{A}}}) = f(\widehat{A}) \subseteq \widehat{B} \subseteq \widehat{A}$. Local Monotonicity, then, implies that $f(\widehat{B}) \subseteq f(\widehat{A})$. We conclude that $\mathcal{C}(A) \subseteq \mathcal{C}(B)$.

The next remark will be useful.

Lemma 5 $\overline{\widehat{A}} \subseteq \mathcal{C}(A)$.

Proof: Since, by Contraction, $f(\widehat{A}) \subseteq \widehat{A}$. ∎

We shall prove Conditional Monotonicity and Threshold Monotonicity. Since $\widehat{A \cup B} \subseteq \widehat{A}$, Coherence implies that we have $\widehat{A \cup B} \cap f(\widehat{A}) \subseteq f(\widehat{A \cup B})$. By Lemma 4, $\widehat{A \cup B} \cap \widehat{\mathcal{C}(A)} \subseteq f(\widehat{A \cup B})$. Therefore $\mathrm{Mod}(A \cup B \cup \mathcal{C}(A)) \subseteq f(\widehat{A \cup B})$ and

$$\mathcal{C}(A, B) \subseteq \overline{\mathrm{Mod}(A \cup B \cup \mathcal{C}(A))} = \overline{\mathrm{Mod}(\mathcal{C}(A) \cup B)} \subseteq \overline{\mathrm{Mod}(\mathcal{C}(A) \cup B \cup C)}, \forall C.$$

By Lemma 5, then, $\mathcal{C}(A, B) \subseteq \mathcal{C}(\mathcal{C}(A), B, C)$. Conditional Monotonicity is obtained by taking $C = \emptyset$. For Threshold Monotonicity, take $A = \mathcal{C}(D)$ and use Idempotence.

We have proved the soundness part. Let us proceed to the proof of completeness: property 1 implies property 2. Assume that $\mathcal{C}$ satisfies Inclusion, Idempotence, Cautious Monotonicity, Conditional Monotonicity and Threshold Monotonicity. We shall now describe $\mathcal{M}$ and the satisfaction relation $\models$ in a unsurprising way. We shall take $\mathcal{M}$ to be set of all sets T of formulas such that $T = \mathcal{C}(T)$. Such a set T will be called a theory. We shall define $\models$ by: $T \models a$ iff $a \in T$.

Let us draw some consequences from the definition of $\models$.

Lemma 6 *For any $A \subseteq \mathcal{L}$, $\widehat{A}$ is the set of all theories that include A and $\overline{\widehat{A}} = \bigcap_{B \subseteq \mathcal{L}} \mathcal{C}(A, B)$.*

Proof: By definition, $\widehat{A}$ is the set of all theories that include A and $\overline{\widehat{A}}$ is the intersection of all theories that include A. Let T be any theory that includes A: $A \subseteq T = \mathcal{C}(T) = \mathcal{C}(A, T)$ and therefore $T = \mathcal{C}(A, B)$ for some B. We have shown that $\bigcap_{B \subseteq \mathcal{L}} \mathcal{C}(A, B) \subseteq \overline{\widehat{A}}$.

But, $\mathcal{C}(A, B)$ is a theory, by Idempotence, and it includes A by Inclusion. Therefore $\overline{\widehat{A}} \subseteq \bigcap_{B \subseteq \mathcal{L}} \mathcal{C}(A, B)$ and our claim is proved. ∎

To simplify notations we shall write:

$$\mathcal{C}n(A) = \overline{\widehat{A}} = \bigcap_{B \subseteq \mathcal{L}} \mathcal{C}(A, B). \tag{9}$$

Since $\mathcal{C}n(A)$ is the intersection of the sets $\mathcal{C}(B)$ for all sets B including A, it is the largest monotonic sub-mapping of $\mathcal{C}$. Since $\mathcal{C}n(A) = \overline{\widehat{A}}$, it is a consequence operation, i.e., satisfies Inclusion, Idempotence and Monotonicity. This remark is very close to a result of J. Dietrich in [6]. Notice that, by Conditional Monotonicity and Threshold Monotonicity we have:

$$\mathcal{C}(A, B) \subseteq \mathcal{C}n(\mathcal{C}(A), B). \tag{10}$$

Lemma 7 *$A \subseteq \mathcal{C}n(A) \subseteq \mathcal{C}(A)$.*

Proof: We have seen that $\mathcal{C}n$ satisfies Inclusion. By taking, in the definition of $\mathcal{C}n$, $B = \emptyset$, one sees that $\mathcal{C}n(A) \subseteq \mathcal{C}(A)$. ∎

Lemma 8 (Right Absorption) *$\mathcal{C}(\mathcal{C}n(A)) = \mathcal{C}(A)$.*

Proof: By Lemmas 7 and 1. ∎

Lemma 9 (Left Absorption) *$\mathcal{C}n(\mathcal{C}(A)) = \mathcal{C}(A)$.*

Proof: By Lemma 7, we have $\mathcal{C}(A) \subseteq \mathcal{C}n(\mathcal{C}(A)) \subseteq \mathcal{C}(\mathcal{C}(A))$. By Idempotence, we have $\mathcal{C}(A) = \mathcal{C}n(\mathcal{C}(A)) = \mathcal{C}(\mathcal{C}(A))$. ▮

We may now define the choice function f. Consider an arbitrary definable set $X = \widehat{A} \subseteq \mathcal{M}$. By Lemma 8, $\widehat{A} = \widehat{B}$ implies $\mathcal{C}(A) = \mathcal{C}(B)$ and therefore we may define $f(X)$ by:

$$f(\widehat{A}) = \widehat{\mathcal{C}(A)}. \tag{11}$$

One immediately sees that the choice function f preserves definability, since $\widehat{\mathcal{C}(A)}$ is definable. We must now show that f satisfies Contraction, Coherence and Local Monotonicity, and that

$$\mathcal{C}(A) = \overline{f(\widehat{A})}. \tag{12}$$

Let us deal with this last question first. By (11), $f(\widehat{A}) = \widehat{\mathcal{C}(A)}$. Therefore $\overline{f(\widehat{A})} = \overline{\widehat{\mathcal{C}(A)}} = \mathcal{C}n(\mathcal{C}(A))$. By Lemma 9, $\overline{f(\widehat{A})} = \mathcal{C}(A)$. We have shown that Equation 12 holds.

It is clear from Equation 11 that f satisfies Contraction. Let us prove now that f satisfies Coherence. Assume $X \subseteq Y$. We have $\overline{Y} \subseteq \overline{X}$ and therefore $\mathcal{C}(\overline{X}) = \mathcal{C}(\overline{Y}, \overline{X})$. By Equation 10, we have $\mathcal{C}(\overline{X}) \subseteq \mathcal{C}n(\mathcal{C}(\overline{Y}), \overline{X})$. Therefore, $\mathrm{Mod}(\mathcal{C}(\overline{Y}) \cup \overline{X}) \subseteq \mathrm{Mod}(\mathcal{C}(\overline{X}))$ and $\widehat{\mathcal{C}(\overline{Y})} \cap \widehat{\overline{X}} \subseteq \widehat{\mathcal{C}(\overline{X})}$. But, $X \subseteq \widehat{\overline{X}}$ and therefore

$$X \cap f(Y) = X \cap Y \cap \widehat{\mathcal{C}(\overline{Y})} \subseteq \widehat{\overline{X}} \cap \widehat{\mathcal{C}(\overline{Y})} \subseteq \widehat{\mathcal{C}(\overline{X})}.$$

We conclude that $X \cap f(Y) \subseteq X \cap \widehat{\mathcal{C}(\overline{X})} = f(X)$.

Finally, let us show that f satisfies Local Monotonicity. Assume $f(\widehat{\overline{X}}) \subseteq Y \subseteq X$. We have, by Equation 12, $\overline{X} \subseteq \overline{Y} \subseteq \overline{f(\widehat{\overline{X}})} = \mathcal{C}(\overline{X})$. By Cautious Monotonicity, then, we conclude that $\mathcal{C}(\overline{X}) \subseteq \mathcal{C}(\overline{Y})$ and $\widehat{\mathcal{C}(\overline{Y})} \subseteq \widehat{\mathcal{C}(\overline{X})}$. Since $Y \subseteq X$, $Y \cap \widehat{\mathcal{C}(\overline{Y})} \subseteq X \cap \widehat{\mathcal{C}(\overline{X})}$. The proof of Theorem 4 is now complete. ▮

One may check that, if the operation $\mathcal{C}$ is monotonic, then, the operation f defined in the construction above is the identity on definable sets: $f(\widehat{A})$ is $\widehat{\mathcal{C}(A)}$, the set of all theories that include $\mathcal{C}(A)$, which is equal to $\widehat{A}$, the set of all theories that include A.

8 Properties of nonmonotonic operations

8.1 First Properties

We shall consider some properties of the operations that satisfy Inclusion, Idempotence, Cautious Monotonicity, Conditional Monotonicity and Threshold Monotonicity, *nonmonotonic* operations in the sequel.

For such operations, the intersection of two theories is not always a theory. Consider, for example, the language that contains two elements a and b and the operation defined by: $\mathcal{C}(A) = A$ for any $A \neq \emptyset$ and $\mathcal{C}(\emptyset) = \{a\}$. This is not a monotonic operation: the only breach of monotonicity is $\mathcal{C}(\emptyset) \not\subseteq \mathcal{C}(b)$. It is easy to check that it satisfies our conditions. But $\mathcal{C}(a) \cap \mathcal{C}(b) = \emptyset$ is not a theory.

In the monotonic framework, the operation $\mathcal{C}$ is defined by the set of its theories, $\mathcal{C}(A)$ being the intersection of all theories including A. In the nonmonotonic framework, this is not the case. Two different operations may define the same set of theories. The example above will prove our case. The operation $\mathcal{C}'$ defined by $\mathcal{C}'(\emptyset) = \{b\}$ and otherwise $\mathcal{C}'(X) = \mathcal{C}(X)$ is different from $\mathcal{C}$ but it has exactly the same theories as $\mathcal{C}$.

The property concerning the intersection of a family of operations holds, but its proof is more intricate that in the monotonic case. If $\mathcal{C}_i$ is a family of nonmonotonic operations (i.e., satisfying Inclusion, Idempotence, Cautious Monotonicity, Conditional Monotonicity and Threshold Monotonicity), its intersection is a nonmonotonic operation. The easiest proof may be semantic: each $\mathcal{C}_i$ is defined by a set $\mathcal{M}_i$ and a choice function f_i. Consider the set $\bigcup_i \mathcal{M}_i$ (assume the $\mathcal{M}_i$ have pairwise empty intersections) and the function f that operates as f_i on $\mathcal{M}_i$ and takes the union of the sets obtained this way. It is easy to see it satisfies Inclusion, Coherence and Local Monotonicity. The operation defined this way is the intersection of the $\mathcal{C}_i$'s.

It is easy to see that, if $\mathcal{C}$ is nonmonotonic, i.e., satisfies Inclusion, Idempotence, Cautious Monotonicity, Conditional Monotonicity and Threshold Monotonicity, then, so is $\mathcal{C}'$ defined by: $\mathcal{C}'(A) = \mathcal{C}(A, B)$.

8.2 Connectives

We shall now find elegant properties of (nonmonotonic) $\mathcal{C}$ that characterize the (semantically) classical propositional connectives. We shall see that the nonmonotonic logic of the (semantically) classical connectives is *classical* propositional logic. We do not consider, in this work, the many possibilities offered by nonmonotonic logics for the definition of non-classical connectives, i.e., connectives that are not defined by truth tables. As in the case of monotonic logic, some additional compactness property is needed for the completeness result. The property required is weak and does not amount at all to requiring the operation $\mathcal{C}$ to be compact.

A word of caution about terminology is needed here. In sequent calculus presentations, connectives are characterized by left and right rules. In natural deduction presentations, they are characterized by introduction and elimination rules. We mix those two terminologies freely to name the rules we are interested in.

Let us consider, first, the case of conjunction. Assume that $\mathcal{L}$ is closed under a binary $\wedge$ and that we consider only satisfaction relations that satisfy:

$$x \models a \wedge b \text{ iff } x \models a \text{ and } x \models b. \tag{13}$$

Then $\mathcal{C}$, defined by $\mathcal{C}(A) = \overline{f(\widehat{A})}$ satisfies:

$$\mathcal{C}(A, a \wedge b) = \mathcal{C}(A, a, b). \tag{14}$$

The proof is very easy: the models that satisfy A and $a \wedge b$ are exactly those that satisfy A, a and b. Notice that the treatment of conjunction is exactly the same as in the monotonic case.

Let us, now, consider the case of negation. Assume $\mathcal{L}$ is closed under a unary $\neg$ and that we consider only satisfaction relations that satisfy:

$$x \models \neg a \text{ iff } x \not\models a. \tag{15}$$

The right introduction rule of the monotonic case: $a \in \mathcal{C}(A)$ implies $\mathcal{C}(A, \neg a) = \mathcal{L}$ is not valid in our nonmonotonic framework. The rules we propose to characterize negation are the following.

$$\mathcal{C}(A, a, \neg a) = \mathcal{L} \tag{16}$$

$$\mathcal{C}(A, \neg a) = \mathcal{L} \Rightarrow a \in \mathcal{C}(A) \tag{17}$$

The validity of those rules is easy to prove. For the first one, there are no models that satisfy $A \cup \{a\} \cup \{\neg a\}$, therefore, by Contraction we have $f(\mathrm{Mod}(A \cup \{a\} \cup \{\neg a\})) = \emptyset$ and $\mathcal{C}(A, a, \neg a) = \mathcal{L}$. For the second one, since no model satisfies all formulas, if $\mathcal{C}(A, \neg a) = \mathcal{L}$, it must be the case that

$f(\mathrm{Mod}(A \cup \{\neg a\})) = \emptyset$. By Coherence, $\mathrm{Mod}(\{\neg a\}) \cap f(\widehat{A}) = \emptyset$. Therefore $f(\widehat{A}) \subseteq \mathrm{Mod}(\{a\})$ and $a \in \mathcal{C}(A)$. One may notice that a similar result fails in Relevance Logic [29], where the *Boolean negation* of [26], defined by (15), is not reasonable.

Consider, now, the case of disjunction. Assume $\mathcal{L}$ is closed under a binary $\vee$ and that we consider only satisfaction relations that satisfy:

$$x \models a \vee b \text{ iff either } x \models a \text{ or } x \models b. \tag{18}$$

The Or introduction rule of monotonic sequent calculus, a left introduction rule, is valid:

$$\mathcal{C}(A, a) \cap \mathcal{C}(A, b) \subseteq \mathcal{C}(A, a \vee b). \tag{19}$$

This follows easily from the fact that a model that satisfies $a \vee b$ satisfies at least one of a or b and from Coherence:

$$f(\mathrm{Mod}(A \cup \{a \vee b\})) \subseteq f(\widehat{A} \cap \widehat{\{a\}}) \cup f(\widehat{A} \cap \widehat{\{b\}}).$$

The right elimination rule of the monotonic case: $\mathcal{C}(A, a \vee b) \subseteq \mathcal{C}(A, a) \cap \mathcal{C}(A, b)$ is not valid. We replace it by a right introduction rule.

$$a \in \mathcal{C}(A) \;\Rightarrow\; a \vee b \in \mathcal{C}(A),\; b \in \mathcal{C}(A) \;\Rightarrow\; a \vee b \in \mathcal{C}(A). \tag{20}$$

Its validity is obvious.

Lastly, consider the case of material implication. Assume $\mathcal{L}$ is closed under a binary $\rightarrow$ and that we consider only satisfaction relations that satisfy:

$$x \not\models a \rightarrow b \text{ iff } x \models a \text{ and } x \not\models b. \tag{21}$$

The right introduction rule of the monotonic case is valid:

$$b \in \mathcal{C}(A, a) \;\Rightarrow\; a \rightarrow b \in \mathcal{C}(A). \tag{22}$$

Assume $b \in \mathcal{C}(A, a)$. Let B be set of all models in $f(\widehat{A})$ that satisfy a. By Coherence, $B \subseteq f(\mathrm{Mod}(A \cup \{a\}))$ and therefore all models in B satisfy b and also $a \rightarrow b$. All models in $f(\widehat{A}) - B$ satisfy $a \rightarrow b$. We conclude that all models in $f(\widehat{A})$ satisfy $a \rightarrow b$ and $a \rightarrow b \in \mathcal{C}(A)$. The right elimination rule of the monotonic case: $a \rightarrow b \in \mathcal{C}(A)$ implies $b \in \mathcal{C}(A, a)$ is not valid. We shall use the following left introduction rule:

$$b \in \mathcal{C}(A, a, a \rightarrow b) \tag{23}$$

which is easily seen to be valid.

Suppose $\mathcal{L}$ is closed under some subset of the connectives $\wedge$, $\neg$, $\vee$ and $\rightarrow$. Is it the case that any operation $\mathcal{C}$ that satisfies Inclusion, Idempotence, Cautious Monotonicity, Conditional Monotonicity, Threshold Monotonicity and the two properties corresponding to each one of the connectives concerned may be defined by a set of models, a satisfaction relation that satisfies the requirement concerning each of the connectives considered and a choice function that satisfies Inclusion, Coherence and Local Monotonicity?

The answer cannot be positive in general. Although the answer is positive for a language $\mathcal{L}$ that contains conjunction as its sole connective, it is negative as soon as $\mathcal{L}$ contains a negation. A compactness assumption will ensure a positive answer. Without any such assumption, already in the monotonic framework, the result does not hold. Consider the following example. The language $\mathcal{L}$ has an infinite set of atomic propositions and is closed under negation. The monotonic operation $\mathcal{C}$ satisfies $a \in \mathcal{C}(A)$ iff $\mathcal{C}(A, \neg a) = \mathcal{L}$, and therefore we may always remove double negations and is defined (up to removal of double negations) by: $\mathcal{C}(A) = A$ if A is finite and does not contain an atomic proposition and its negation, and $\mathcal{C}(A) = \mathcal{L}$ otherwise (i.e., if A is infinite or contains an atomic proposition and its negation). Notice that this $\mathcal{C}$ fails the Lindenbaum lemma: there are consistent sets but no maximal consistent set. Assume $\mathcal{C}$ is representable by a suitable f. Let $Y = f(\widehat{\emptyset})$ and $y \in Y$. We have $\{y\} = \widehat{\overline{y}}$, and, by Coherence, $y \in f(\{y\})$. Also, $f(\{y\}) = \widehat{\mathcal{C}(\overline{y})}$. But $\overline{y}$ includes a infinite number of atomic propositions or an infinite number of negations of atomic propositions, therefore $\mathcal{C}(\overline{y}) = \mathcal{L}$ and $f(\{y\}) = \emptyset$. A contradiction to $y \in f(\{y\})$. We conclude that $Y = \emptyset$. But then $\mathcal{C}(\emptyset) = \mathcal{L}$, a contradiction.

We shall, then, assume that $\mathcal{C}$ satisfies the following:

Weak Compactness $\quad \mathcal{C}(A) = \mathcal{L} \Rightarrow \exists$ a finite $B \subseteq_f A$ such that $\mathcal{C}(B) = \mathcal{L}$.

Notice that the Weak Compactness assumed does not imply, in the nonmonotonic framework, that, if $a \in \mathcal{C}(A)$, then there is a finite subset B of A such that $a \in \mathcal{C}(B)$, even when proper connectives are available. In the monotonic case, the monotonic left introduction rule for negation makes Compactness follow from Weak Compactness, but we rejected this rule.

Before we state and prove the main theorem of this section, let us build the tools for the proof. We shall assume that $\mathcal{C}$ satisfies Inclusion, Idempotence, Cautious Monotonicity, Conditional Monotonicity, Threshold Monotonicity and Weak Compactness. We assume that $\mathcal{L}$ is closed under some set of connectives that includes negation and that $\mathcal{C}$ satisfies the two properties described above for each of the connectives assumed to be in the language (in particular it satisfies 16 and 17). Recall that A is a theory iff $A = \mathcal{C}(A)$.

Definition 3 *A set $A \subseteq \mathcal{L}$ is said to be* inconsistent *iff $\mathcal{C}(A) = \mathcal{L}$. A set that is not inconsistent is said to be* consistent.

Notice that, by Idempotence, A is consistent iff $\mathcal{C}(A)$ is consistent, and that there is only one inconsistent theory, namely $\mathcal{L}$.

Lemma 10 *If $A \subseteq B$ and A is inconsistent, then B is inconsistent.*

Proof: We have: $A \subseteq B \subseteq \mathcal{L} = \mathcal{C}(A)$. By Cautious Monotonicity, then, $\mathcal{L} = \mathcal{C}(A) \subseteq \mathcal{C}(B)$. ▮

The following notion is fundamental.

Definition 4 *A set A is said to be* maximal consistent *iff it is consistent and any strict superset $B \supset A$ is inconsistent.*

The next two lemmas are central.

Lemma 11 *If A is consistent, there is some maximal consistent set B such that $A \subseteq B$.*

Proof: The proof is as in the classical case. It is included only for completeness sake. Consider any ascending chain of consistent sets, A_i, $i \in I$, where $i < j$ implies $A_i \subseteq A_j$. We claim that the union $B = \bigcup_{i \in I} A_i$ is consistent. If it were inconsistent, by Weak Compactness, there would be some finite inconsistent subset of B. This subset would be a subset of A_i for some i, and by Lemma 10, A_i would be inconsistent, contrary to assumption. We have shown that the union of any ascending chain of consistent sets is consistent. Zorn's lemma, then, implies that any consistent A may be embedded in a maximal consistent set. ▮

Lemma 12 *If A is maximal consistent, then*

1. *A is a theory,*

2. *(if $\wedge$ is in the language) $a \wedge b \in A$ iff $a \in A$ and $b \in A$,*

3. *$\neg a \in A$ iff $a \notin A$,*

4. *(if $\vee$ is in the language) $a \vee b \in A$ iff $a \in A$ or $b \in A$,*

5. *(if $\rightarrow$ is in the language) $a \rightarrow b \notin A$ iff $a \in A$ and $b \notin A$.*

Proof: Assume A is maximal consistent. If $\mathcal{C}(A)$ were a strict superset of A it would be inconsistent, but then A would be inconsistent. Therefore $\mathcal{C}(A) = A$.

By 14, and since A is a theory, $a \wedge b \in A$ iff $a \in A$ and $b \in A$. The maximality of A is not used here.

Assume $\neg a \in A$. If we had $a \in A$, we would, by 16 have $\mathcal{C}(A) = \mathcal{L}$, a contradiction to the consistency of A. Assume, now, $\neg a \notin A$. Since A is maximal consistent, $\mathcal{C}(A, \neg a) = \mathcal{L}$. By 17, we have $a \in \mathcal{C}(A) = A$.

Assume $a \in A$ or $b \in A$. By the right introduction rule 20, $a \vee b \in \mathcal{C}(A) = A$. Assume, now that $a \vee b \in A$. We must show that at least one of a or b is in A. Suppose, a contrario, that neither one is in A. Since A is maximal consistent, both $\mathcal{C}(A, a)$ and $\mathcal{C}(A, b)$ are equal to $\mathcal{L}$. But, then, the left introduction rule 19 implies $\mathcal{C}(A) = \mathcal{C}(A, a \vee b) = \mathcal{L}$, a contradiction.

Assume $a \notin A$. Since A is maximal consistent, $\mathcal{C}(A, a) = \mathcal{L}$, and therefore $b \in \mathcal{C}(A, a)$ and, by the right introduction rule 22, $a \rightarrow b \in \mathcal{C}(A) = A$. Suppose, now, that $a \in A$. If $a \rightarrow b \in A$, then, by the left introduction rule 23, $b \in \mathcal{C}(A) = A$. If $a \rightarrow b \notin A$, then, by the right introduction rule 22, $b \notin \mathcal{C}(A, a) = \mathcal{C}(A) = A$. ∎

The next theorem shows that the Introduction Elimination rules above define exactly the (semantically) classical connectives.

Theorem 5 *Assume that $\mathcal{L}$ is closed under some of the propositional connectives, including negation, and that $\mathcal{C}$ satisfies the Introduction and Elimination properties described above for each of the connectives of the language. Let $\mathcal{C}$ satisfy Inclusion, Idempotence, Cautious Monotonicity, Conditional Monotonicity, Threshold Monotonicity and Weak Compactness. Then, there is a set $\mathcal{M}$, a satisfaction relation $\models$* that behaves classically *for each of the existing connectives and a definability-preserving choice function f that satisfies Contraction, Coherence and Local Monotonicity that defines $\mathcal{C}$, i.e., such that $\mathcal{C}(A) = \overline{f(\widehat{A})}$, for any $A \subseteq \mathcal{L}$.*

Proof: The proof proceeds exactly as the proof of the completeness part of Theorem 4, except that, for the set $\mathcal{M}$ we take, not all theories, but only the maximal consistent sets of formulas. The proof proceeds exactly in the same way, as soon as we have proved Lemma 13 to replace Lemma 6. The fact that the satisfaction relation behaves classically for the connectives follows from Lemma 12. ∎

Lemma 13 *For any $A \subseteq \mathcal{L}$, $\widehat{A}$ is the set of all maximal consistent sets that include A and $\overline{\widehat{A}} = \bigcap_{B \subseteq \mathcal{L}} \mathcal{C}(A, B)$.*

Proof: By definition, $\widehat{A}$ is the set of all maximal consistent sets that include A and $\overline{\widehat{A}}$ is the intersection of all maximal consistent sets that include A. Let T be such a set. By Lemma 12, T is a theory and: $A \subseteq T = \mathcal{C}(T) = \mathcal{C}(A, T)$ and therefore $T = \mathcal{C}(A, B)$ for some B. We have shown that $\bigcap_{B \subseteq \mathcal{L}} \mathcal{C}(A, B) \subseteq \overline{\widehat{A}}$.

But, suppose, now, that $a \not\in \mathcal{C}(A, B)$. By 17, $\mathcal{C}(A, B, \neg a) \neq \mathcal{L}$ (we need here the assumption that negation is in the language). By Lemma 11, $\mathcal{C}(A, B, \neg a)$ is a subset of some maximal consistent set. This set, since it is consistent and contains $\neg a$, does not contain a (see 16). Therefore $a \not\in \overline{\widehat{A}}$. We have shown that $\overline{\widehat{A}} \subseteq \bigcap_{B \subseteq \mathcal{L}} \mathcal{C}(A, B)$ and our claim is proved. ∎

One may ask whether the result holds even for languages that do not include negation. The question is open. But, notice that, in the proof, we make use of the fact that, if T is a theory and $a \not\in T$, then there is a maximal consistent superset of T that does not include a. This does not hold in general: for example, if $\mathcal{C}$ is the operation of deduction of intuitionistic logic and p is an atomic proposition, $p \not\in \mathcal{C}(\neg\neg p)$, but any maximal consistent set that includes $\neg\neg p$ must include p.

We may now show that propositional nonmonotonic logic is not weaker than (and therefore exactly the same as) monotonic logic.

Theorem 6 *Let $\mathcal{L}$ be a propositional calculus and $a, b \in \mathcal{L}$. The following propositions are equivalent.*

1. *a logically implies b, i.e., $a \models b$,*
2. *for any operation $\mathcal{C}$ that satisfies Inclusion, Idempotence, Monotonicity, Weak Compactness and the Introduction-Elimination rules above: $b \in \mathcal{C}(a)$,*
3. *for any operation $\mathcal{C}$ that satisfies Inclusion, Idempotence, Cautious Monotonicity, Conditional Monotonicity and Threshold Monotonicity, Weak Compactness and the Introduction-Elimination rules above: $b \in \mathcal{C}(a)$,*
4. *for any such $\mathcal{C}$ and for any $A \subseteq \mathcal{L}$: $b \in \mathcal{C}(A, a)$,*

5. *for any such $\mathcal{C}$: $\mathcal{C}(a, \neg b) = \mathcal{L}$.*

Proof: Property 5 implies 4, since, by Cumulativity, $\mathcal{C}(a, \neg b) = \mathcal{L}$ implies $\mathcal{C}(A, a, \neg b) = \mathcal{L}$, and, by the Left Elimination rule for negation: $b \in \mathcal{C}(A, a)$. Property 4 obviously implies 3, that obviously implies 2. It is easy to see that property 2 implies 1. Let m be any propositional model that satisfies a. Let $\mathcal{C}$ be defined by $\mathcal{C}(A) = \overline{\{m\}}$, the set of formulas satisfied by m, if $m \in \widehat{A}$ and $\mathcal{C}(A) = \mathcal{L}$ otherwise. By assumption, $b \in \mathcal{C}(a)$. But $\mathcal{C}(a) = \overline{\{m\}}$ since $m \in \widehat{\{a\}}$, therefore $m \models b$.

The only non-trivial part of the proof is that 1 implies 5. Assume $a \models b$ and $\mathcal{C}$ satisfies Inclusion, Idempotence, Cautious Monotonicity, Conditional Monotonicity and Threshold Monotonicity, Weak Compactness and the Introduction-Elimination rules. By Theorem 5, there is a set $\mathcal{M}$, a satisfaction relation $\models$ that behaves classically with respect to the connectives and a definability-preserving choice function satisfying Contraction, Coherence and Local Monotonicity such that $\mathcal{C}(a, \neg b) = \overline{f(\widehat{\{a\}} \cap \widehat{\{\neg b\}})}$. But, by assumption $\widehat{\{a\}} \cap \widehat{\{\neg b\}} = \emptyset$. By Contraction, then $\mathcal{C}(a, \neg b) = \overline{\emptyset} = \mathcal{L}$. ∎

Theorem 6 shows that the proof theory of the semantically-classical propositional connectives in a nonmonotonic setting is the same as in a monotonic setting. The nonmonotonic setting is very rich, and it is tempting to consider, there, connectives the semantics of which is not locally truth-functional: the truth-value of $\Box a$ or of $a \succ b$ in a model m depending on the choice function f. This is left for future work.

It is customary to consider Introduction-Elimination rules as definitions of the connectives. Hacking [17, Section VII] discusses this idea and proposes that, to be considered as bona fide definitions of the connectives, the rules must be such that they ensure that any legal logic on a small language may be conservatively extended to a legal logic on the language extended by closure under the connective. We may ask whether any nonmonotonic operation $\mathcal{C}$ on $\mathcal{L}$ that satisfies Inclusion, Idempotence, Cautious Monotonicity, Conditional Monotonicity and Threshold Monotonicity may be conservatively extended to such an operation that satisfies the Introduction-Elimination rules above: (14), (16), (17), (19), (20), (22) and 23). From the discussion above, just before the definition of Weak Compactness, it seems that Weak Compactness will be required. The question of whether Weak Compactness is sufficient to ensure a conservative extension is open. The result will be proven under a stronger assumption: the set of atomic propositions is finite (this essentially the Simplifying Assumption of Section 5.1).

Theorem 7 *Let $\mathcal{L}$ be the propositional calculus on a* finite *set P of atomic propositions. Let $\mathcal{C}$ be an operation on P that satisfies Inclusion, Idempotence, Cautious Monotonicity, Conditional Monotonicity and Threshold Monotonicity. Then, there exists an operation $\mathcal{C}$' on $\mathcal{L}$ that satisfies Inclusion, Idempotence, Cautious Monotonicity, Conditional Monotonicity, Threshold Monotonicity and the rules: (14), (16), (17), (19), (20), (22) and (23), such that, for any $A \subseteq P$, $\mathcal{C}(A) = P \cap \mathcal{C}'(A)$.*

Proof: Since P is finite, $\mathcal{C}$ is trivially weakly compact, the assumptions of Theorem 5 hold and $\mathcal{C}$ is therefore generated by some $\mathcal{M}$, $\models$ and f. All subsets of M are definable and f is therefore defined on all subsets of M. We may extend $\models$ to the language $\mathcal{L}$ by using equations (13), (15), (18) and(21). The choice function f, then defines an operation $\mathcal{C}$' on $\mathcal{L}$ by (1). Since f satisfies Contraction, Coherence and Local Monotonicity, the operation $\mathcal{C}$' satisfies Inclusion, Idempotence, Cautious Monotonicity, Conditional Monotonicity and Threshold Monotonicity. Since the models of M satisfy equations (13), (15), (18) and(21), $\mathcal{C}$ satisfies the Introduction-Elimination rules. It is left to us to see that $\mathcal{C}(A) = P \cap \mathcal{C}'(A)$, for any $A \subseteq P$. This follows straightforwardly from the fact that both $\mathcal{C}(A)$ and $\mathcal{C}'(A)$ are the set of formulas (the former of P, the latter of $\mathcal{L}$) satisfied by all models of the set $f(\widehat{A})$. ∎

9 Comparison with previous work

Let us assume that $\mathcal{L}$ is a propositional language, and that $\mathcal{C}$ satisfies Weak Compactness, Inclusion, Idempotence, Cautious Monotonicity, Conditional Monotonicity, Threshold Monotonicity and the Introduction-Elimination rules for all the propositional connectives. One may define a consequence operation on $\mathcal{L}$ by: $a \mid\!\sim b$ iff $b \in \mathcal{C}(\{a\})$.

Theorem 8 *Under the assumptions above, the relation $\mid\!\sim$ is preferential (see [19]).*

Proof: The result follows easily from Theorem 5. We shall treat only two of the six properties, the reader will easily treat the other four properties. Consider Left Logical Equivalence. Assume that a is logically equivalent to a'. Then $\widehat{\{a\}} = \widehat{\{a'\}}$ and $\mathcal{C}(\{a\}) = \mathcal{C}(\{a'\})$. Consider **Or**. Assume $c \in \mathcal{C}(\{a\})$ and $c \in \mathcal{C}(\{b\})$. Any $x \in f(\widehat{\{a\}})$ satisfies c. Any $x \in f(\widehat{\{b\}})$ satisfies c. Any

f satisfying Contraction and Coherence, satisfies $f(X \cup Y) \subseteq f(X) \cup f(Y)$. Therefore any $x \in f(\widehat{\{a\}}) \cup f(\widehat{\{b\}})$ satisfies c and $c \in \mathcal{C}(a \vee b)$. ▮

Does the converse hold, i.e., may any preferential relation be obtained from such an operation $\mathcal{C}$ in such a way? The answer is yes.

Theorem 9 *Let $\mid\!\sim$ be any preferential relation. There is an operation $\mathcal{C}$ satisfying all the assumptions above such that $a \mid\!\sim b$ iff $b \in \mathcal{C}(\{a\})$.*

Proof: This follows from the construction of Theorem 14 of [13]. The theorem claims that any finitary preferential operation (i.e., preferential relation) may be conservatively extended to an infinitary preferential operation $\mathcal{C}$. The definition of $\mathcal{C}$ is the following: $b \in \mathcal{C}(A)$ iff there exists some formula a such that $A \models a$ ($\models$ is logical implication of propositional calculus) enjoying the following property: for any a' such that $A \models a'$ and $a' \models a$, one has $a' \mid\!\sim b$.

The reader may check, with no need to use the theorem of [13] and relatively easily, that $\mathcal{C}$ satisfies all the properties requested. For example, $\mathcal{C}$ is weakly compact. Assume $\mathcal{C}(A) = \mathcal{L}$. Then **false** $\in \mathcal{C}(A)$ and there is some a such that $A \models a$ and $a \mid\!\sim$**false**. But there is a finite subset B of A such that $B \models a$ and $B \mid\!\sim$**false**. Therefore **false** $\in \mathcal{C}(B)$. ▮

10 Rational Monotonicity

In this section, an important sub-family of nonmonotonic operations will be described. It corresponds, in the present setting, to the rational relations of [22]. The qualitative measures of Section 5.3 were partial orders. Measures (probability measures, for example) provide orders that obey an additional modularity property:

$$X > Y,\ X \not> Z \Rightarrow Z > Y. \tag{24}$$

This seems a very natural property to require of a qualitative measure. Assume for example that there is a function $m : 2^{\mathcal{M}} \longrightarrow \mathbf{R}$, the set of real numbers such that $X > Y$ iff $m(X) > m(Y)$, then, $>$ is modular.

What is the property of choice functions that correspond to the modularity of a qualitative measure?

$$\textbf{Arrow}\quad X \subseteq Y,\ X \cap f(Y) \neq \emptyset \ \Rightarrow\ f(X) \subseteq f(Y)$$

This property is in fact only one half of the property studied by K. Arrow in [3], assuming Contraction. The other half is Coherence. Note indeed that Arrow, Coherence and Contraction imply

$$X \subseteq Y,\ X \cap f(Y) \neq \emptyset \ \Rightarrow \ f(X) = X \cap f(Y), \tag{25}$$

the property originally considered by K. Arrow. Indeed, by Coherence $X \cap f(Y) \subseteq f(X)$; by Arrow, $f(X) \subseteq f(Y)$; by contraction $f(X) \subseteq X$.

The intuitive justification for Arrow is some kind of laziness principle for the choice function: if $X \subseteq Y$ and we have already a list of the best elements of Y, we shall take for the best elements of X exactly those best elements of Y that happen to be in X, at least whenever this new list is not empty. A remark will now show that there is a natural family of choice functions that satisfy Arrow. Suppose the elements of $\mathcal{M}$ are ranked: our real estate agent, for example, gives a grade to every available apartment and, when asked about apartments in some area, delivers the list of all available apartments in this area *that have the highest ranking.* If the highest ranking in Paris is 10 and one of those apartments graded 10 happens to be on the left bank, then all best apartments on the left bank have grade 10 and are therefore part of the list of best apartments in Paris. We shall now show that Modularity and Arrow are indeed exact counterparts.

Theorem 10 *If $>$ satisfies properties 4 and 24, then, the choice function f defined by taking for $f(X)$ the set of heavy elements (see Definition 2) of X satisfies Arrow. If f is a choice function that satisfies Contraction, Coherence, Local Monotonicity and Arrow, then, the relation $>$ defined by: $X > Y$ iff $f(X) \neq \emptyset$ and $Y \cap f(X \cup Y) = \emptyset$ satisfies property 24.*

Proof: Assume $>$ satisfies properties 4 and 24, and that $X \subseteq Y$ and $x \in X$ such that $Y \not> \{x\}$. Let x' be any heavy element of X: $X \not> \{x'\}$. If x' was not heavy in Y, we would have $Y > \{x'\}$ and, by 24, $Y > X$ and, by 4, $Y > \{x\}$, a contradiction. Therefore any heavy element of X is a heavy element of Y.

Assume now that f satisfies Contraction, Coherence, Local Monotonicity and Arrow and that $X > Y$. We shall show that either $X > Z$ or $Z > Y$. We know that $f(X) \neq \emptyset$ and $Y \cap f(X \cup Y) = \emptyset$. We distinguish two cases. Assume, first, that $(Y \cup Z) \cap f(X \cup Y \cup Z) \neq \emptyset$. By Arrow, then, $f(Y \cup Z) \subseteq f(X \cup Y \cup Z)$. But, by Contraction and then Coherence

$$f(X \cup Y \cup Z) = f(X \cup Y \cup (Z - Y)) =$$

$(X\cup Y)\cap f(X\cup Y\cup(Z-Y))\cup(Z-Y)\cap f(X\cup Y\cup(Z-Y))\subseteq f(X\cup Y)\cup(Z-Y)$.

Therefore $Y\cap f(Y\cup Z)=\emptyset$. If $f(Z)\neq\emptyset$, then, we have $Z>Y$. If $f(Z)=\emptyset$, since, by Coherence, $Z\cap f(X\cup Z)\subseteq f(Z)$, we have $X>Z$.

We are left with the case $(Y\cup Z)\cap f(X\cup Y\cup Z)=\emptyset$. By Contraction, $f(X\cup Y\cup Z)\subseteq X\subseteq X\cup Z\subseteq X\cup Y\cup Z$. By Local Monotonicity, then, $f(X\cup Z)\subseteq f(X\cup Y\cup Z)$. Therefore $Z\cap f(X\cup Z)=\emptyset$ and we conclude that $X>Z$. ▮

Having proved the equivalence of Modularity and Arrow, we shall prove equivalence Modularity and the property of Rational Monotonicity for the operation $\mathcal{C}$.

Theorem 11 *Suppose we are given a language $\mathcal{L}$ and a function $\mathcal{C}: 2^{\mathcal{L}}\longrightarrow 2^{\mathcal{L}}$. Then, the following two conditions are equivalent:*

1. *$\mathcal{C}$ satisfies Inclusion, Idempotence, Cautious Monotonicity, Conditional Monotonicity, Threshold Monotonicity and*

   Rational Monotonicity $\mathcal{C}(\mathcal{C}(A),B)\neq\mathcal{L}\ \Rightarrow\ \mathcal{C}(A)\subseteq\mathcal{C}(A,B)$,

2. *there exists a set $\mathcal{M}$ (of models), a satisfaction relation $\models\subseteq\mathcal{M}\times\mathcal{L}$ and a definability-preserving choice function $f:\mathcal{M}\longrightarrow\mathcal{M}$ satisfying Contraction, Coherence, Local Monotonicity and Arrow such that $\mathcal{C}(A)=\overline{f(\widehat{A})}$.*

Rational Monotonicity, introduced in [19], has been studied at length in [22, 13]. Its intuitive justification is that, when the new information, contained in B, is consistent with what had been concluded from A, $\mathcal{C}(A)$, the new information will not force us to retract any previous conclusions, it may only add new conclusions. This is also a laziness principle.

Proof: We shall describe only those parts of the proof that differ from that of Theorem 4. For the soundness part: 2 implies 1, define $\mathcal{C}(A)$ as $\overline{f(\widehat{A})}$ and assume Arrow. We must derive Rational Monotonicity.

Assume $\mathcal{C}(\mathcal{C}(A),B)\neq\mathcal{L}$. On one hand we have $\mathrm{Mod}(A\cup B)\subseteq\widehat{A}$ and on the other hand we have $\overline{f(\mathrm{Mod}(\mathcal{C}(A)\cup B))}\neq\mathcal{L}$. Therefore, $f(\mathrm{Mod}(\mathcal{C}(A)\cup B))\neq\emptyset$ and, by Contraction, $\mathrm{Mod}(\mathcal{C}(A)\cup B)\neq\emptyset$, in other terms, $\widehat{\mathcal{C}(A)}\cap\widehat{B}\neq\emptyset$. Since f preserves definability: $\widehat{\mathcal{C}(A)}=f(\widehat{A})$. We see that $f(\widehat{A})\cap\widehat{B}\neq\emptyset$ and, by Contraction, $f(\widehat{A})\cap\mathrm{Mod}(A\cup B)\neq\emptyset$. By Arrow, $f(\mathrm{Mod}(A\cup B))\subseteq f(\widehat{A})$. Therefore $\mathcal{C}(A)\subseteq\mathcal{C}(A,B)$.

For the completeness part, we define $\mathcal{C}n$ as in the proof of Theorem 4. The properties of $\mathcal{C}n$ proved there hold true. We must define $\mathcal{M}$ slightly more carefully. We take for elements of $\mathcal{M}$ only those sets $T \neq \mathcal{L}$ such that $T = \mathcal{C}(T)$. The satisfaction relation is defined as previously: $T \models a$ iff $a \in T$. We see that, by construction, no element of $\mathcal{M}$ satisfies all formulas of $\mathcal{L}$ and therefore $\overline{X} = \mathcal{L}$ implies $X = \emptyset$. We define f as previously. The proof that Equation 12 holds is unchanged. Assuming Rational Monotonicity, we must show that Arrow holds. Assume $X \subseteq Y$ and $X \cap f(Y) \neq \emptyset$. We have $\widehat{\overline{X}} \subseteq \widehat{\overline{Y}}$ and $X \cap f(Y) = X \cap Y \cap \widehat{\mathcal{C}(\overline{Y})} = X \cap \widehat{\mathcal{C}(\overline{Y})} \neq \emptyset$. Therefore $\widehat{\overline{X}} \cap \widehat{\mathcal{C}(\overline{Y})} \neq \emptyset$. Let us define $A = \overline{X}$ and $B = \overline{B}$. On one hand, we have $\widehat{A} \subseteq \widehat{B}$ and therefore $\mathcal{C}n(B) \subseteq \mathcal{C}n(A)$ and on the other hand we have $\widehat{A} \cap \widehat{\mathcal{C}(B)} \neq \emptyset$. Therefore $\mathrm{Mod}(A \cup \mathcal{C}(B)) \neq \emptyset$. By the remark made at the start of this proof, when paying attention to exclude $\mathcal{L}$ from the set $\mathcal{M}$, we see that $\overline{\mathrm{Mod}(A \cup \mathcal{C}(B))} \neq \mathcal{L}$. Therefore $\mathcal{C}(A, \mathcal{C}(B)) \neq \mathcal{L}$. It is easy to show that $\mathcal{C}(A, \mathcal{C}(B)) = \mathcal{C}(\mathcal{C}n(A), \mathcal{C}(\mathcal{C}n(B)))$. We may therefore use Rational Monotonicity to conclude that we have $\mathcal{C}(B) \subseteq \mathcal{C}(A)$, i.e., $\mathcal{C}(\overline{Y}) \subseteq \mathcal{C}(\overline{X})$ and $\widehat{\mathcal{C}(\overline{X})} \subseteq \widehat{\mathcal{C}(\overline{Y})}$. Therefore $X \cap \widehat{\mathcal{C}(\overline{X})} \subseteq Y \cap \widehat{\mathcal{C}(\overline{Y})}$, i.e., $f(X) \subseteq f(Y)$. ∎

We have shown that the additional property of Rational Monotonicity, studied in the literature, corresponds exactly to an additional property of the choice function f. The family of nonmonotonic operations satisfying Rational Monotonicity has different closure properties than the larger family studied in the preceding sections. In particular, it is not closed under intersection. The reader will easily find a counter example. One such example is provided in [25].

11 Conclusion

We have described two quite different but equivalent semantic frameworks: choice functions and qualitative measures, that provide an ontology for nonmonotonic deduction. Choice functions have been studied by researchers in Social Choice for their *rationalizability* properties, i.e., by what kind of aggregation mechanism can they arise from individual preferences? The equivalence we have shown with qualitative measures may be of interest to those researchers. The family of nonmonotonic operations defined in Section 4 is precisely the family defined by choice functions or by qualitative measures, it is a natural generalization of Tarski's monotonic deductive operations.

The operations of this family are closed under intersection. The classical connectives may be defined elegantly for this family of operations, by properties that are weaker than those generally considered in the monotonic case. Only a very mild compactness assumption is needed. The sentential connectives may be *defined* by Introduction-Elimination rules. The connectives defined have a classical semantics. A further property of choice functions, considered by K. Arrow [3], is shown to be equivalent to the modularity (i.e., negative-transitivity) of the qualitative measure and the operations defined are characterized by the additional property of Rational Monotonicity of [19, 22].

A large number of questions and alleys for future research are left open by this work. Let us mention a few, roughly from the small and technical to the vast and philosophical.

The equivalence of Qualitative Measures and Choice functions semantics has been shown only under the Simplifying Assumption. A more general equivalence requires the introduction of a family of definable sets in the framework of Qualitative Measures and probably the introduction of some counterpart to Definability Preservation.

Theorem 7 has been proved only under the assumption that P is finite. Does it hold without this restriction, and if not, does it hold without this restriction if one assumes $\mathcal{C}$ to be weakly compact?

The framework of Choice Functions begs the semantic definition of non-classical, non-truth functional connectives. The study of such connectives (unary or, more probably, binary) in nonmonotonic logics seems particularly exciting. The *preferred* interpretation of the choice function suggests a link with deontic logics.

This work sheds new light on properties studied by researchers in Social Choice. In particular new insights on the case of an infinite set of outcomes have been presented. Are they relevant to the Social Choice community?

In[22], a positive answer was given to a question that can now be seen as equivalent to: given a choice function that satisfies Contraction, Coherence and Local Monotonicity, is there a canonical way to restrict this choice function in a way that ensures the Arrow property? This canonical construction, rational closure, offers a way to aggregate individual preferences into collective preferences that satisfy the Arrow property. This aggregation method does not satisfy Independence from Irrelevant Alternatives. Is it of any interest for Social Choice?

This work uses Tarski's framework. Most proof-theoretic studies use

Gentzen's framework. The translation of the results of this paper to the language of Gentzen's sequents may be illuminating. It seems one will have to consider sequents whose sides may be infinite sets of formulas.

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