Deductive Nonmonotonic Inference Operations: Antitonic Representations *

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October 4, 1999

Abstract

We provide a characterization of those nonmonotonic inference operations $\mathcal{C}$ for which $\mathcal{C}(X)$ may be described as the set of all logical consequences of X together with some set of additional assumptions $\mathcal{S}(X)$ that depends anti-monotonically on X (i.e., $X \subseteq Y$ implies $\mathcal{S}(Y) \subseteq \mathcal{S}(X)$). The operations represented are exactly characterized in terms of properties most of which have been studied in [3]. Similar characterizations of right-absorbing and cumulative operations are also provided. For cumulative operations, our results fit in closely with those of [2]. We then discuss extending finitary operations to infinitary operations in a canonical way and discuss co-compactness properties. Our results provide a satisfactory notion of pseudo-compactness, generalizing to deductive nonmonotonic operations the notion of compactness for monotonic operations. They also provide an alternative, more elegant and more general, proof of the existence of an infinitary deductive extension for any finitary deductive operation (Theorem 7.9 of [3]).

*This work was partially supported by the Jean and Helene Alfassa fund for research in Artificial Intelligence

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1 Introduction

A fundamental intuition of *Default Reasoning* (understood in the wider sense) is that a reasoner has, at its disposal, a set of facts X (a fact is represented by a formula) and a set of defaults D (there is no general agreement on the way the defaults ought to be represented). Given those, it draws conclusions (conclusions are formulas, as facts are) by extracting from D and X some set of assumptions A (formulas again), held in the presence of X, and then accepting as plausible conclusions the set of logical consequences of the set $X \cup A$. Some nonmonotonic systems are explicitly presented in this way, most notably the Closed World Assumption of Reiter's [10], the system described by Poole in [9] (at least in its skeptical version when the set of defaults is finite), the system based on Epistemic Entrenchment [4], and the Rational Closure construction of [6]. Other systems, most notably the Default Logic of [11] and the Circumscription of [7] are presented, at first, in ways that do not fit this paradigm, but they could have been presented this way: Default Logic adds to the facts the conclusions of the applicable defaults, and Circumscription adds to the facts what can be deduced from the defaults when abnormalities are minimized. In fact any system may be presented in this way by taking for A the set of plausible conclusions from X.

In such a presentation, it seems very natural to expect that, for any fixed set of defaults D, the mapping $\mathcal{S}$ that sends a set of facts X to the set of assumptions $A = \mathcal{S}(X)$ held in the presence of X, be antitonic (i.e., anti-monotonic, i.e., $X \subseteq Y$ implies $\mathcal{S}(Y) \subseteq \mathcal{S}(X)$). Indeed, this is explicitly the case for the Closed World Assumption. For finite Poole systems, even those without constraints, this is not the case, though. There, given a set D of formulas, $\mathcal{C}(X)$ is defined as $\mathcal{C}n(X, \bigcap_{B \in \mathcal{B}(X)} \mathcal{C}n(B))$, where $\mathcal{B}(X)$ is the set of all subsets of D that are consistent with X and maximal for this property. The intersection must not be antitonic. Intuitively, antitonicity is a very natural property since $\mathcal{S}(X)$ is some set of default assumptions that are *compatible*, or consistent with X, and the larger X is, the less compatible (with X) formulas there are. Antitonicity seems to be a necessary requirement for $\mathcal{S}$, at least when X has a single *extension*. The purpose of this paper is to characterize those operations that may be defined by such an antitonic representation. From our characterization, will follow that many nonmonotonic systems that were not originally presented in such a way are amenable to an antitonic presentation. This point will be taken up in the conclusion. In particular finite Poole systems without constraints have an antitonic representation, even though, as explained above, their natural presentation is not antitonic.

In [1], S. Brass considers a property (IMD, Definition 3.13) that may seem closely related to the antitonicity of $\mathcal{S}$. In fact, IMD is very different from it. In our notations, IMD may be described as: if $x \in \mathcal{C}(\emptyset)$ and $X \subseteq Y$, then if x is in $\mathcal{C}(Y)$ it is also in $\mathcal{C}(X)$. There does not seem to be much intuitive support for such a property.

We suppose a language $\mathcal{L}$ is given, and, with it some consequence operation $\mathcal{C}n$ in the sense of Tarski. The elements of $\mathcal{L}$ will be referred to as formulas. About $\mathcal{C}n$, we shall assume, as customary in the literature, that $\mathcal{C}n$ satisfies inclusion, monotonicity, idempotency and compactness. As usual, we write $\mathcal{C}n(X, Y)$ instead of $\mathcal{C}n(X \cup Y)$, and $X \models Y$ for $Y \subseteq \mathcal{C}n(X)$. We shall assume that the language $\mathcal{L}$ has *implication*. i.e., for any formulas $a, b \in \mathcal{L}$, there is a formula $a \rightarrow b$ such that, for any $X \subseteq \mathcal{L}$, $b \in \mathcal{C}n(X, a)$ iff $a \rightarrow b \in \mathcal{C}n(X)$. Some of our results do not depend on this assumption, or could be proved with weaker assumptions on $\mathcal{L}$. We shall use the following lemma from [3, Lemma A.3].

Lemma 1

1. *For any set Y and any finite set A of formulas, there is a set $A \rightarrow Y$ of formulas such that, for any set X of formulas, $A \rightarrow Y \subseteq \mathcal{C}n(X)$ iff $Y \subseteq \mathcal{C}n(X, A)$. If Y is finite, so is $A \rightarrow Y$.*

2. *For any $X, Y, Z \subseteq \mathcal{L}$, $\mathcal{C}n(X, Y) \cap \mathcal{C}n(X, Z) = \mathcal{C}n(X, \mathcal{C}n(Y) \cap \mathcal{C}n(Z))$.*

3. *For any finite set A of formulas and any family $Y_i, i \in I$ of sets of formulas, $\mathcal{C}n(A, \bigcap_{i \in I} \mathcal{C}n(Y_i)) = \bigcap_{i \in I} \mathcal{C}n(A, Y_i)$.*

2 Infinitary operations

We consider infinitary operations $\mathcal{C} : 2^{\mathcal{L}} \longrightarrow 2^{\mathcal{L}}$. We shall need to consider a host of properties for such operations. Most of them were considered in [3]. Our terminology is slightly different, since we consider operations that are not always cumulative, or even absorbing. After the definitions, we shall compare those properties of infinitary operations with the corresponding properties of finitary consequence relations described in [5].

The following have to be understood for arbitrary subsets X and Y of $\mathcal{L}$, and an arbitrary element x of $\mathcal{L}$. We shall write $A \subseteq_f X$ if A is a finite subset of X.

$$\text{(supraclassicality)} \qquad \mathcal{C}n(X) \subseteq \mathcal{C}(X) \tag{1}$$

$$\text{(left absorption)} \qquad \mathcal{C}n(\mathcal{C}(X)) = \mathcal{C}(X) \tag{2}$$

$$\text{(right absorption)} \qquad \mathcal{C}n(X) = \mathcal{C}n(Y) \;\Rightarrow\; \mathcal{C}(X) = \mathcal{C}(Y) \tag{3}$$

$$\text{(deductivity)} \qquad Y \subseteq X \;\Rightarrow\; \mathcal{C}(X) \subseteq \mathcal{C}n(X, \mathcal{C}(Y)) \tag{4}$$

$$\text{(cumulativity)} \qquad Y \subseteq \mathcal{C}n(\mathcal{C}(X)) \;\Rightarrow\; \mathcal{C}n(\mathcal{C}(X, Y)) = \mathcal{C}n(\mathcal{C}(X)) \tag{5}$$

$$\text{(antitonicity)} \qquad X \subseteq Y \;\Rightarrow\; \mathcal{C}(Y) \subseteq \mathcal{C}(X) \tag{6}$$

$$\text{(compactness)} \qquad x \in \mathcal{C}(X) \;\Rightarrow\; \exists A_x \subseteq_f X \text{ such that } \forall Y, \text{ if } A_x \subseteq_f Y \subseteq X, \text{ then } x \in \mathcal{C}(Y) \tag{7}$$

$$\text{(supracompactness)} \qquad x \in \mathcal{C}(X) \;\Rightarrow\; \exists A_x \subseteq_f X \text{ such that } \forall Y, \text{ if } A_x \subseteq_f Y \subseteq \mathcal{C}(X), \text{ then } x \in \mathcal{C}(Y) \tag{8}$$

Supraclassicality is a consequence of Reflexivity and Right-Weakening. Left-absorption corresponds to Right-Weakening + And. Right-absorption corresponds to Left-Logical-Equivalence. The central property of this paper, Deductivity, corresponds to the (S) rule. To understand its intuitive appeal, consider the case $X = Y \cup \{a\}$ (a is an arbitrary formula). Deductivity says that, if $x \in \mathcal{C}(Y, a)$, then x should already be in $\mathcal{C}n(a, \mathcal{C}(Y))$, i.e., that $a \rightarrow x$ should be in $\mathcal{C}(Y)$. In other words, if x is expected to be true on the evidence "a and Y", then, on the evidence Y alone, we should already expect "if a then x". Cumulativity corresponds to Cut + Cautious Monotonicity.

Right-absorption could have been written as $\mathcal{C}(X) = \mathcal{C}(\mathcal{C}n(X))$. We have chosen a formulation that generalizes without change to finitary operations. Cumulativity is also defined in a slightly more convoluted way than usual since we intend to consider cumulative operations that do not satisfy left-absorption. The definition of supracompactness is taken from [2]. Our generalization of compactness is different both from the compactness considered in [2] and from supracompactnes. It is clear that any supracompact operation is compact (our meaning), and that any compact operation in our sense is compact in Freund's sense. For monotonic operations our notion of compactness coincides with the usual one. For supraclassical, left-absorbing, cumulative operations, we shall see in Corollary 1 that compact operations are exactly the supracompact operations of Freund. Our first result characterizes those operations that have an antitonic representation of the sort we discussed in the introduction. The gist of our theorem is that, assuming supraclassicality and left-absorption, the existence of an antitonic representation is essentially equivalent to deductivity plus compacity. Since compacity is of concern only for infinitary operations, for finitary operations antitonic representations are equivalent to the (S) rule.

Theorem 1 *Let $\mathcal{C}$ be an infinitary operation. The following three properties are equivalent.*

1. *There is an antitonic operator $\mathcal{S}$ such that, for any set X of formulas, $\mathcal{C}(X) = \mathcal{C}n(X, \mathcal{S}(X))$,*
2. *for any set X,*
$$\mathcal{C}(X) = \mathcal{C}n(X, \bigcap_{Y \subseteq X} \mathcal{C}(Y)), \tag{9}$$
3. *$\mathcal{C}$ is supraclassical, left-absorbing, deductive and compact.*

Proof: We show, first, that 1 implies 3. Let $\mathcal{C}(X) = \mathcal{C}n(X, \mathcal{S}(X))$, for some antitonic $\mathcal{S}$. The operation $\mathcal{C}$ is supraclassical because $\mathcal{C}n$ satisfies inclusion and monotonicity. It is left-absorbing because $\mathcal{C}n$ is idempotent. Let us check it is deductive. Suppose $Y \subseteq X$. Then $\mathcal{S}(X) \subseteq \mathcal{S}(Y)$, since $\mathcal{S}$ is antitonic, and by monotonicity of $\mathcal{C}n$,

$$\mathcal{C}(X) = \mathcal{C}n(X, \mathcal{S}(X)) \subseteq \mathcal{C}n(X, \mathcal{S}(Y)) \subseteq \mathcal{C}n(X, \mathcal{C}n(Y, \mathcal{S}(Y))) = \mathcal{C}n(X, \mathcal{C}(Y)).$$

Let us show it is also compact. Suppose $x \in \mathcal{C}(X)$. By the compactness of $\mathcal{C}n$, there is a finite set $B_x \subseteq_f X \cup \mathcal{S}(X)$ such that $x \in \mathcal{C}n(B_x$. Let $A_x \stackrel{\text{def}}{=} B_x \cap X$. We have $A_x \subseteq_f X$, and, by monotonicity of $\mathcal{C}n$,

$$\mathcal{C}n(B_x) = \mathcal{C}n(A_x, B_x \cap \mathcal{S}(X)) \subseteq \mathcal{C}n(A_x, \mathcal{S}(X)),$$

and $x \in \mathcal{C}n(A_x, \mathcal{S}(X))$. If $A_x \subseteq_f Y \subseteq X$, then $x \in \mathcal{C}n(Y, \mathcal{S}(Y))$ by monotonicity of $\mathcal{C}n$ and antitonicity of $\mathcal{S}$.

We now show that 3 implies 2. Suppose $\mathcal{C}$ is supraclassical, left-absorbing, deductive and compact. Since, $\bigcap_{Y \subseteq X} \mathcal{C}(Y) \subseteq \mathcal{C}(X)$ and $\mathcal{C}$ is supraclassical and left-absorbing, we see easily that

$$\mathcal{C}n(X, \bigcap_{Y \subseteq X} \mathcal{C}(Y)) \subseteq \mathcal{C}(X).$$

Suppose, now, that $x \in \mathcal{C}(X)$. Since $\mathcal{C}$ is compact, there is a finite set $A \subseteq_f X$, such that $x \in \mathcal{C}(Y)$, for any Y, $A \subseteq_f Y \subseteq X$. In other terms, $x \in \mathcal{C}(A, Y)$, for any $Y \subseteq X$. Since $\mathcal{C}$ is deductive

$$x \in \bigcap_{Y \subseteq X} \mathcal{C}n(A, Y, \mathcal{C}(Y)) = \bigcap_{Y \subseteq X} \mathcal{C}n(A, \mathcal{C}(Y)).$$

By Lemma 1, part 3,

$$x \in \mathcal{C}n(A, \bigcap_{Y \subseteq X} \mathcal{C}(Y)) \subseteq \mathcal{C}n(X, \bigcap_{Y \subseteq X} \mathcal{C}(Y)).$$

The fact that 2 implies 1 is obvious since $\mathcal{S}(X) \stackrel{\text{def}}{=} \bigcap_{Y \subseteq X} \mathcal{C}(Y)$ is antitonic. ▮

It is clear that there are supraclassical, left-absorbing deductive operations that are not compact. Consider for example the operation $\mathcal{C}$ defined by $\mathcal{C}(X) = \mathcal{C}n(X, a)$ if X is infinite and $\mathcal{C}(X) = \mathcal{C}n(X)$ otherwise. The antitonic operator that appears in Equation (9), i.e. $\bigcap_{Y \subseteq X} \mathcal{C}(Y)$, is the largest antitonic operator that represents $\mathcal{C}$, as will be explained now.

Lemma 2 *If, for some antitonic $\mathcal{S}$, for any $X \subseteq \mathcal{L}$, $\mathcal{C}(X) = \mathcal{C}n(X, \mathcal{S}(X))$ then, $\mathcal{S}(X) \subseteq \bigcap_{Y \subseteq X} \mathcal{C}(Y)$.*

Proof: Let Y be an arbitrary subset of X. We have, since $\mathcal{S}$ is antitonic,

$$\mathcal{S}(X) \subseteq \mathcal{S}(Y) \subseteq \mathcal{C}n(Y, \mathcal{S}(Y)) = \mathcal{C}(Y).$$

▮

Our next result extends Theorem 1 to right-absorbing operations. In [2] the intersection of Equation 10 was called the trace of X.

Theorem 2 *Let $\mathcal{C}$ be an infinitary operation. The following three properties are equivalent.*

1. *There is an antitonic right-absorbing operator $\mathcal{S}$ such that, for any set X of formulas, $\mathcal{C}(X) = \mathcal{C}n(X, \mathcal{S}(X))$,*

2. for any set X,

$$\mathcal{C}(X) = \mathcal{C}n(X, \bigcap_{X \models Y} \mathcal{C}(Y)), \tag{10}$$

3. $\mathcal{C}$ is supraclassical, left-absorbing, right-absorbing, deductive and compact.

Proof: Let us show, first, that 1 implies 3. This is clear from Theorem 1 and the fact that, if $\mathcal{S}$ is right-absorbing, so is $\mathcal{C}n(X, \mathcal{S}(X))$. We now show that 3 implies 2. From right-absorption of $\mathcal{C}$ and Theorem 1, we have

$$\mathcal{C}(X) = \mathcal{C}(\mathcal{C}n(X)) = \mathcal{C}n(\mathcal{C}n(X), \bigcap_{X \models Y} \mathcal{C}(Y))$$

and we conclude easily. The last leg of the proof is obvious since

$$\mathcal{S}(X) \stackrel{\text{def}}{=} \bigcap_{X \models Y} \mathcal{C}(Y)$$

is antitonic and right-absorbing. ∎

Notice that we do not claim that, for a right-absorbing $\mathcal{C}$ the intersections appearing in Equations (9) and (10) are equal. The following example will show it need not be the case. Consider the propositional calculus on two variables p and q, and let $\mathcal{C}n$ be logical consequence. Define $\mathcal{S}(X)$ as $\mathcal{C}n(p)$ if $\mathcal{C}n(X) = \mathcal{C}n(\emptyset)$ and $\emptyset$ otherwise. The operator $\mathcal{S}$ is obviously antitonic and right-absorbing. If $\mathcal{C}(X) \stackrel{\text{def}}{=} \mathcal{C}n(X, \mathcal{S}(X))$ one sees that p is an element of $\mathcal{C}(\emptyset)$ and of $\mathcal{C}(p)$, but not an element of $\mathcal{C}(q \rightarrow p)$. Therefore $p \in \bigcap_{Y \subseteq \{p\}} \mathcal{C}(Y)$ but $p \notin \bigcap_{p \models Y} \mathcal{C}(Y)$. Similarly, to what we have shown in Lemma 2, the antitonic operation, $\bigcap_{X \models Y} \mathcal{C}(Y)$ is the largest right-absorbing antitonic representation of $\mathcal{C}$.

Lemma 3 *If $\mathcal{C}(X) = \mathcal{C}n(X, \mathcal{S}(X))$ for some antitonic right-absorbing $\mathcal{S}$, then, $\mathcal{S}(X) \subseteq \bigcap_{X \models Y} \mathcal{C}(Y)$*

Proof: Let Y be an arbitrary subset of $\mathcal{C}n(X)$. We have, since $\mathcal{S}$ is right-absorbing and antitonic,

$$\mathcal{S}(X) = \mathcal{S}(\mathcal{C}n(X)) \subseteq \mathcal{S}(Y) \subseteq \mathcal{C}n(Y, \mathcal{S}(Y)) = \mathcal{C}(Y).$$

▮

We deal now with cumulative operations. Our first result is important in itself and will be used in the proof of our third characterization result. Notice, first, that any supraclassical, left-absorbing cumulative operation is right-absorbing, since $\mathcal{C}n(X) \subseteq \mathcal{C}(X)$ implies $\mathcal{C}n(\mathcal{C}(X, \mathcal{C}n(X)) = \mathcal{C}n(\mathcal{C}(X))$ and therefore $\mathcal{C}(\mathcal{C}n(X)) = \mathcal{C}(X)$.

Theorem 3 *If $\mathcal{C}$ is supraclassical, left-absorbing, deductive and cumulative, then*

$$\bigcap_{Y \subseteq \mathcal{C}n(X)} \mathcal{C}(Y) = \bigcap_{Y \subseteq \mathcal{C}(X)} \mathcal{C}(Y). \tag{11}$$

Proof: Since $\mathcal{C}$ is supraclassical, the right-hand side is obviously a subset of the left-hand side. We must show that, for any $Z \subseteq \mathcal{C}(X)$,

$$\bigcap_{Y \subseteq \mathcal{C}n(X)} \mathcal{C}(Y) \subseteq \mathcal{C}(Z).$$

But, since $\mathcal{C}n(X) \cap \mathcal{C}n(Z) \subseteq \mathcal{C}n(X)$, $\bigcap_{Y \subseteq \mathcal{C}n(X)} \mathcal{C}(Y) \subseteq \mathcal{C}(\mathcal{C}n(X) \cap \mathcal{C}n(Z))$, and it is enough to show that $\mathcal{C}(\mathcal{C}n(X) \cap \mathcal{C}n(Z)) \subseteq \mathcal{C}(Z)$. We notice that, since $\mathcal{C}$ is left-absorbing and supraclassical, $\mathcal{C}n(X) \cap \mathcal{C}n(Z) \subseteq \mathcal{C}(X)$. But $\mathcal{C}$ is left-absorbing and cumulative and we conclude that $\mathcal{C}(X) = \mathcal{C}(X, \mathcal{C}n(X) \cap \mathcal{C}n(Z))$ and therefore $Z \subseteq \mathcal{C}(X, \mathcal{C}n(X) \cap \mathcal{C}n(Z))$. But $\mathcal{C}$ is deductive and

$$\mathcal{C}(X, \mathcal{C}n(X) \cap \mathcal{C}n(Z)) \subseteq \mathcal{C}n(X, \mathcal{C}n(X) \cap \mathcal{C}n(Z), \mathcal{C}(\mathcal{C}n(X) \cap \mathcal{C}n(Z)))$$

and we conclude that $Z \subseteq \mathcal{C}n(X, \mathcal{C}(\mathcal{C}n(X) \cap \mathcal{C}n(Z)))$. But, obviously,

$$Z \subseteq \mathcal{C}n(Z, \mathcal{C}(\mathcal{C}n(X) \cap \mathcal{C}n(Z))).$$

By Lemma 1, part 2,

$$\mathcal{C}n(X, \mathcal{C}(\mathcal{C}n(X) \cap \mathcal{C}n(Z))) \cap \mathcal{C}n(Z, \mathcal{C}(\mathcal{C}n(X) \cap \mathcal{C}n(Z)))$$

$$= \mathcal{C}n(X \cap Z, \mathcal{C}(\mathcal{C}n(X) \cap \mathcal{C}n(Z))) = \mathcal{C}(\mathcal{C}n(X) \cap \mathcal{C}n(Z)).$$

The last equality follows from supraclassicality and left-absorption. We conclude that $Z \subseteq \mathcal{C}(\mathcal{C}n(X) \cap \mathcal{C}n(Z))$. By the cumulativity of $\mathcal{C}$, we have $\mathcal{C}(\mathcal{C}n(X) \cap \mathcal{C}n(Z)) = \mathcal{C}(\mathcal{C}n(X) \cap \mathcal{C}n(Z), Z)$. But $\mathcal{C}n(\mathcal{C}n(X) \cap \mathcal{C}n(Z), Z) = \mathcal{C}n(Z)$ and, since $\mathcal{C}$ is right-absorbing by a remark above, $\mathcal{C}(\mathcal{C}n(X) \cap \mathcal{C}n(Z)) = \mathcal{C}(Z)$.
∎

We may now prove our third characterization theorem.

Theorem 4 *Let $\mathcal{C}$ be an infinitary operation. The following three properties are equivalent.*

1. *There is an antitonic, right-absorbing and cumulative operator $\mathcal{S}$ such that, for any set X of formulas, $\mathcal{C}(X) = \mathcal{C}n(X, \mathcal{S}(X))$,*
2. *for any set X,*
$$\mathcal{C}(X) = \mathcal{C}n(X, \bigcap_{Y \subseteq \mathcal{C}(X)} \mathcal{C}(Y)), \tag{12}$$
3. *$\mathcal{C}$ is supraclassical, left-absorbing, deductive, cumulative and compact.*

Proof: Let us show, first, that 1 implies 3. Let $\mathcal{C}$ be as in 1. By Theorem 2, we only need to show that $\mathcal{C}$ is cumulative. Suppose, therefore, that we have $Y \subseteq \mathcal{C}n(\mathcal{C}(X)) = \mathcal{C}n(X, \mathcal{S}(X))$. One of the inclusions we have to prove (the one corresponding to Cut) is a consequence of the deductivity of $\mathcal{C}$ (using $X \subseteq X \cup Y$) and does not require the cumulativity of $\mathcal{S}$. Indeed $\mathcal{C}(X, Y) \subseteq \mathcal{C}n(X, Y, \mathcal{C}(X))$ by the deductivity of $\mathcal{C}$. Since both X and Y are subsets of $\mathcal{C}(X)$ and $\mathcal{C}$ is left-absorbing, we conclude that $\mathcal{C}(X, Y) \subseteq \mathcal{C}(X)$. The converse inclusion will be proved now. Since $Y \subseteq \mathcal{C}n(X, \mathcal{S}(X))$, for any $y \in Y$, there exists a finite $A_y \subseteq_f X$ such that $y \in \mathcal{C}n(A_y, \mathcal{S}(X))$. We may apply part 1 of Lemma 1, and consider the set $W \stackrel{\text{def}}{=} \{A_y \rightarrow y \mid y \in Y\} \subseteq \mathcal{C}n(\mathcal{S}(X))$. But $\mathcal{S}$ is cumulative and therefore, $\mathcal{C}n(\mathcal{S}(X)) = \mathcal{C}n(\mathcal{S}(X, W))$. One easily sees that $\mathcal{C}n(X, W) = \mathcal{C}n(X, Y)$, and, since $\mathcal{S}$ is right-absorbing, we have $\mathcal{C}n(\mathcal{S}(X)) = \mathcal{C}n(\mathcal{S}(X, Y))$. Therefore

$$\mathcal{C}(X) = \mathcal{C}n(X, \mathcal{S}(X)) = \mathcal{C}n(X, \mathcal{S}(X, Y)) \subseteq \mathcal{C}n(X, Y, \mathcal{S}(X, Y)) = \mathcal{C}(X, Y).$$

Let us show now that 3 implies 2. Suppose $\mathcal{C}$ is supraclassical, left-absorbing, deductive, cumulative and compact. As we noticed above, any supraclassical, left-absorbing and cumulative operation is right-absorbing.

We may use Theorem 2 to see that Equation 10 holds, and conclude by Theorem 3. Finally, let us show that 2 implies 1. Suppose $\mathcal{C}$ satisfies Equation 12. We shall show that $\mathcal{S}(X) \stackrel{\text{def}}{=} \bigcap_{Y \subseteq \mathcal{C}(X)} \mathcal{C}(Y)$ is antitonic, right-absorbing and cumulative. None of those properties is obvious. One immediately sees that $\mathcal{C}$ is supraclassical and left-absorbing. One easily sees that $\mathcal{C}$ is deductive. Indeed, suppose $Y \subseteq X$, by supraclassicality we have $Y \subseteq \mathcal{C}(X)$ and therefore $\mathcal{S}(X) \subseteq \mathcal{C}(Y)$. We conclude that $\mathcal{C}(X) = \mathcal{C}n(X, \mathcal{S}(X)) \subseteq \mathcal{C}n(X, \mathcal{C}(Y))$. The crux of the proof is to show that $\mathcal{C}$ is cumulative. Let $Z \subseteq \mathcal{C}(X)$ (remember $\mathcal{C}$ is left-absorbing). Since $\mathcal{C}$ is deductive, we have $\mathcal{C}(X, Z) \subseteq \mathcal{C}n(Z, \mathcal{C}(X)) = \mathcal{C}(X)$. But, in turn, this implies $\mathcal{S}(X) \subseteq \mathcal{S}(X, Z)$, and

$$\mathcal{C}(X) = \mathcal{C}n(X, \mathcal{S}(X)) \subseteq \mathcal{C}n(X, Z, \mathcal{S}(X, Z)) = \mathcal{C}(X, Z).$$

We have shown that $\mathcal{C}$ is supraclassical, left-absorbing, deductive and cumulative. By Theorem 3, we conclude that $\mathcal{S}(X) = \bigcap_{X \models Y} \mathcal{C}(Y)$ and we conclude that $\mathcal{S}$ is antitonic and right-absorbing. The only thing left to prove is that $\mathcal{S}$ is cumulative. Suppose $Z \subseteq \mathcal{C}n(\mathcal{S}(X))$. We see that $Z \subseteq \mathcal{C}(X)$, and, since $\mathcal{C}$ is cumulative, we have $\mathcal{C}(X) = \mathcal{C}(X, Z)$. We conclude that $\mathcal{S}(X) = \mathcal{S}(X, Z)$. ▮

The following corollary will make completely clear the relation between compacity and supracompacity.

Corollary 1 *Let $\mathcal{C}$ be supraclassical, left-absorbing, deductive and cumulative. It is supracompact iff it is compact.*

Proof: The *only if* part is obvious from the supraclassicality of $\mathcal{C}$ and the definitions. For the *if* part, by Theorem 4, Equation 12 holds. Supracompacity follows easily. ▮

3 Finitary deductive operations

We shall denote the set of all finite subsets of X by $\mathcal{P}_f(X)$. We shall now consider finitary operations $\mathcal{F} : \mathcal{P}_f(\mathcal{L}) \longrightarrow 2^{\mathcal{L}}$. The properties of supraclassicality, left-absorption, right-absorption, deductivity, antitonicity and cumulativity for finitary operations are defined exactly as for infinitary operations, after replacing arbitrary sets X and Y by finite sets A and B. Notice that right-absorption cannot be expressed as $\mathcal{F}(A) = \mathcal{F}(\mathcal{C}n(A))$ since $\mathcal{C}n(A)$ need not be finite. The best way to study finitary operations is to extend them to infinitary operations and use the representation theorems we have developed in the previous section. We shall see that there is a canonical way to extend finitary operations. But, before we can do that, we must prove one technical result concerning finitary operations. Its proof is the corresponding part of the proof of Theorem 1, after replacing arbitrary sets by finite sets wherever needed.

Lemma 4 *Let $\mathcal{F}$ be a supraclassical, left-absorbing and deductive finitary operation. Then, for any finite $A \subseteq_f \mathcal{L}$, $\mathcal{F}(A) = \mathcal{C}n(A, \bigcap_{B \subseteq A} \mathcal{F}(B))$.*

∎

In the next section we shall deal with the question of extending finitary operations.

4 Co-compactness

Our notion of compactness is not really satisfying, since it does not seem to be the *right* generalization of the notion of compactness (for monotonic operations) since any finitary monotonic operation has a unique compact monotonic extension, but compact extensions of nonmonotonic operations are not unique. If we restrict our attention to supraclassical, left-absorbing, deductive and compact operations the new notion we need seems to be the following.

Definition 1 *An operation $\mathcal{C}$ is said to be* strongly co-compact *iff, for any $X \subseteq \mathcal{L}$ and for any $x \in \mathcal{L}$, if $x \notin \mathcal{C}(X)$, then there is a finite $A \subseteq_f X$ such that $x \notin \mathcal{C}(A)$.*

The following is obvious.

Lemma 5 *If $\mathcal{S}$ is antitonic, it is strongly co-compact iff $\mathcal{S}(X) = \bigcap_{A \subseteq_f X} \mathcal{S}(A)$.*

Not all operations represented by antitonic operators are strongly co-compact, as may be seen from the following counter-example. Consider a propositional calculus on an infinite set of propositional variables, and let $\mathcal{C}n$ be logical consequence. Let a a proposition that is not a tautology. Define, for any finite set B, $\mathcal{S}(B) \stackrel{\text{def}}{=} \mathcal{C}n(\{\chi_B \rightarrow a\})$, where χ_B is the conjunction of all elements of B. For infinite X, $\mathcal{S}(X) \stackrel{\text{def}}{=} \bigcap_{B \subseteq_f X} \mathcal{S}(B)$. It is easy to see that $a \in \mathcal{C}(B) = \mathcal{C}n(B, \mathcal{S}(B))$, for any finite set $B \subseteq_f \mathcal{L}$. Let X be any infinite set of propositional variables that do not appear in a. One may see that $\bigcap_{B \subseteq_f X} \mathcal{S}(B)$ is equal to $\mathcal{C}n(\emptyset)$, and therefore $\mathcal{C}(X) = \mathcal{C}n(X)$. We conclude that $a \notin \mathcal{C}(X)$, even though $a \in \mathcal{C}(A)$ for any finite subset A of X. Many operations represented by antitonic operations are strongly co-compact, though.

Theorem 5 *If $\mathcal{C}(X) = \mathcal{C}n(X, \mathcal{S}(X))$ for some antitonic and strongly co-compact $\mathcal{S}$, and if $\mathcal{S}(\emptyset)$ is finite, then $\mathcal{C}$ is strongly co-compact.*

Proof: Since $\mathcal{S}$ is antitonic and strongly co-compact, we have $\mathcal{C}(X) = \mathcal{C}n(X, \bigcap_{B \subseteq_f X} \mathcal{S}(B))$. Since, for any Y, $\mathcal{S}(Y) \subseteq \mathcal{S}(\emptyset)$ and this last set is finite, the intersection above is a finite intersection. By Lemma 1, part 2, $\mathcal{C}(X) = \bigcap_{B \subseteq_f X} \mathcal{C}n(X, \mathcal{S}(B))$. Suppose $x \notin \mathcal{C}(X)$. There is a finite set $B \subseteq_f X$, such that $x \notin \mathcal{C}n(X, \mathcal{S}(B))$. Clearly, $x \notin \mathcal{C}n(B, \mathcal{S}(B)) = \mathcal{C}(B)$. ∎

Our next result shows that compactness and strong co-compactness together play the role of a generalization of the notion of compactness.

Theorem 6 *Let $\mathcal{F}$ be a finitary supraclassical, left-absorbing, deductive operation. It has a unique supraclassical, left-absorbing, deductive, compact and strongly co-compact extension.*

Proof: Suppose $\mathcal{F}$ is a finitary supraclassical, left-absorbing, deductive operation. We shall prove uniqueness of the extension first. Suppose that $\mathcal{C}$ is an infinitary supraclassical, left-absorbing, deductive, compact and strongly co-compact extension of $\mathcal{F}$. By Theorem 1, $\mathcal{C}(X) = \mathcal{C}n(X, \bigcap_{Y \subseteq X} \mathcal{C}(Y))$. But $\mathcal{C}$ is strongly co-compact, and

$$\bigcap_{Y \subseteq X} \mathcal{C}(Y) = \bigcap_{B \subseteq_f X} \mathcal{C}(B).$$

We conclude that

$$\mathcal{C}(X) = \mathcal{C}n(X, \bigcap_{B \subseteq_f X} \mathcal{F}(B)). \tag{13}$$

But Equation (13) uniquely defines $\mathcal{C}$ in terms of $\mathcal{F}$.

To prove existence, we shall show that Equation (13) provides an extension of $\mathcal{F}$ with all the required properties. Lemma 4 shows that it is indeed an extension of $\mathcal{F}$. We notice that Equation (13) provides an antitonic representation of $\mathcal{C}$ and therefore Theorem 1 enables us to conclude that $\mathcal{C}$ is supraclassical, left-absorbing, deductive and compact. It is left to us to show that $\mathcal{C}$ is strongly co-compact. But this follows clearly from Equation (13) since $\bigcap_{B \subseteq_f X} \mathcal{F}(B) \subseteq \mathcal{C}(X)$. ∎

Corollary 2 *If $\mathcal{F}$ is supraclassical, left-absorbing and deductive, its unique supraclassical, left-absorbing, deductive, compact and strongly co-compact extension is given by $x \in \mathcal{C}(X)$ iff there is a finite $A_x \subseteq_f X$ such that $x \in \mathcal{F}(A_x, B)$ for any finite $B \subseteq_f X$.*

Proof: Let $\mathcal{C}'$ be the unique extension of $\mathcal{F}$ provided by Equation (13). One easily checks that $\mathcal{C}$ is also an extension of $\mathcal{F}$, and, since $\mathcal{C}'$ is compact, we have $\mathcal{C}'(X) \subseteq \mathcal{C}(X)$. Suppose now that $x \in \mathcal{C}(X)$. We have $x \in \bigcap_{B \subseteq_f X} \mathcal{F}(A_x, B)$. Since $\mathcal{F}$ is deductive, by Lemma 1 and Equation (13),

$$x \in \bigcap_{B \subseteq_f X} \mathcal{C}n(A_x, B, \mathcal{F}(B)) = \bigcap_{B \subseteq_f X} \mathcal{C}n(A_x, \mathcal{F}(B)) = \mathcal{C}n(A_x, \bigcap_{B \subseteq_f X} \mathcal{F}(B))$$

$$\subseteq \mathcal{C}n(X, \bigcap_{B \subseteq_f X} \mathcal{F}(B)) = \mathcal{C}'(X).$$

∎

5 Right-absorbing operations

We shall now deal with extending right-absorbing finitary operations. It turns out that we cannot prove that any supraclassical, left-absorbing, right-absorbing and deductive finitary operation has a supraclassical, left-absorbing, right-absorbing, deductive, compact and strongly co-compact extension. We, therefore, need a weaker notion of co-compactness.

Definition 2 *An operation $\mathcal{C}$ is said to be* co-compact *iff, for any $X \subseteq \mathcal{L}$ and for any $x \in \mathcal{L}$, if $x \notin \mathcal{C}(X)$, then there is a finite set A, such that $X \models A$ and $x \notin \mathcal{C}(A)$.*

Clearly, any strongly co-compact operation is co-compact. We also have the following, the proof of which is obvious.

Lemma 6 *If $\mathcal{C}$ is strongly co-compact, then the operation $\mathcal{C}'$ defined by $\mathcal{C}'(X) = \mathcal{C}(\mathcal{C}n(X))$ is co-compact.*

We may now prove our result concerning right-absorbing operations.

Theorem 7 *Any supraclassical, left-absorbing, right-absorbing and deductive finitary operation $\mathcal{F}$ has a unique extension that is supraclassical, left-absorbing, right-absorbing, deductive, compact and co-compact.*

Proof: Let $\mathcal{C}$ be the unique supraclassical, left-absorbing, deductive, compact and strongly co-compact extension of $\mathcal{F}$ defined by Equation (13). Consider $\mathcal{C}'$ defined by $\mathcal{C}'(X) = \mathcal{C}(\mathcal{C}n(X))$. The operation $\mathcal{C}'$ is easily seen to be supraclassical, left-absorbing, right-absorbing, deductive and compact. By Lemma 6, it is also co-compact. By Corollary 2, $x \in \mathcal{C}'(X)$ iff there is a finite $A_x \subseteq_f \mathcal{C}n(X)$ such that $x \in \mathcal{F}(A_x, B)$ for any finite $B \subseteq_f \mathcal{C}n(X)$. This shows immediately that $\mathcal{C}'$ is an extension of $\mathcal{F}$. We have proved existence.

For uniqueness notice that, if $\mathcal{C}$ has the properties required, we have $\mathcal{C}(X) = \mathcal{C}n(X, \bigcap_{X \models Y} \mathcal{C}(Y))$, by Theorem 2. But, if $\mathcal{C}$ is a co-compact extension of $\mathcal{F}$, $\bigcap_{X \models Y} \mathcal{C}(Y) = \bigcap_{B \subseteq_f \mathcal{C}n(X)} \mathcal{C}(B)$. ∎

The extension of Theorem 7 is characterized in the following way.

Corollary 3 *If $\mathcal{F}$ is supraclassical, left-absorbing, right-absorbing and deductive, its unique supraclassical, left-absorbing, right-absorbing, deductive, compact and co-compact extension is given by $x \in \mathcal{C}(X)$ iff there is a finite $A_x \subseteq_f X$ such that $x \in \mathcal{F}(A_x, B)$ for any finite $B \subseteq_f \mathcal{C}n(X)$.*

Proof: In the proof of Theorem 7 we have seen this extension is $\mathcal{C}(\mathcal{C}n(X))$ for the extension $\mathcal{C}$ that is described in Corollary 2. Therefore $x \in \mathcal{C}(X)$ iff there is a finite $A_x \subseteq_f \mathcal{C}n(X)$ such that $x \in \mathcal{F}(A_x, B)$ for any finite $B \subseteq_f \mathcal{C}n(X)$. But if $A_x \subseteq_f \mathcal{C}n(X)$, there is a finite $A'_x \subseteq_f X$ such that $A'_x \models A_x$, and therefore $\mathcal{F}(A'_x, B) = \mathcal{F}(A'_x, A_x, B)$ by right-absorption. ∎

Corollary 3 shows that the unique extension of $\mathcal{F}$ is the operation denoted $\mathcal{C}_{\mathcal{F}}$ in [3] (the only case considered there was the case of a cumulative $\mathcal{F}$). Our next result deals with cumulative operations.

Theorem 8 *The unique supraclassical, left-absorbing, right-absorbing, deductive, compact and co-compact extension $\mathcal{C}$ of a supraclassical, left-absorbing, right-absorbing, deductive and cumulative finitary operation $\mathcal{F}$ is cumulative.*

Proof: We have seen in the proof of Theorem 7 that this unique extension is defined by $\mathcal{C}(X) = \mathcal{C}n(X, \bigcap_{X \models B} \mathcal{F}(B))$. By the first leg of Theorem 4, all we have to show is that $\mathcal{S}(X) \stackrel{\rm def}{=} \bigcap_{X \models B} \mathcal{F}(B)$ is cumulative. We immediately see that $\mathcal{S}$ is antitonic. Since $\mathcal{F}$ is left-absorbing, so is $\mathcal{S}$. Suppose, then, that $Y \subseteq \mathcal{S}(X)$. We must show that $\mathcal{S}(X) = \mathcal{S}(X, Y)$. By antitonicity of $\mathcal{S}$, we immediately see that $\mathcal{S}(X, Y) \subseteq \mathcal{S}(X)$. We shall show that $\mathcal{S}(X) \subseteq \mathcal{S}(X, Y)$. Let A be an arbitrary finite set such that $X, Y \models A$, we shall show that $\mathcal{S}(X) \subseteq \mathcal{F}(A)$. Since $X, Y \models A$, for any $a \in A$, there is a finite $Y_a \subseteq_f Y$, such that $X \models Y_a \rightarrow a$. In fact, $Y_a \rightarrow a$ is a singleton. Let $B \stackrel{\rm def}{=} \bigcup_{a \in A} Y_a \rightarrow a$. The set B is finite. Notice that $A \models B$ and $B, Y \models A$. There is therefore a finite subset C of Y such that $B, C \models A$. We see that $\mathcal{C}n(B, C) = \mathcal{C}n(A, C)$. Since $X \models B$, we have, by the definition of $\mathcal{S}$, $\mathcal{S}(X) \subseteq \mathcal{F}(B)$. It will be enough to prove that $\mathcal{F}(B) = \mathcal{F}(A)$. But, indeed, $C \subseteq_f Y \subseteq \mathcal{S}(X) \subseteq \mathcal{F}(B)$ and by the cumulativity of $\mathcal{F}$, $\mathcal{F}(B) = \mathcal{F}(B, C)$. By right-absorption $\mathcal{F}(B, C) = \mathcal{F}(A, C)$. We conclude that $A \subseteq_f \mathcal{F}(B)$, and, by cumulativity of $\mathcal{F}$, $\mathcal{F}(B) = \mathcal{F}(B, A)$, but $\mathcal{C}n(B, A) = \mathcal{C}n(A)$, and we conclude that $\mathcal{F}(B) = \mathcal{F}(A)$. ▮

Theorem 8 represents an improvement on Theorem 7.9 of [3], in which the language $\mathcal{L}$ was assumed to have a disjunction. There, it was shown that, under the additional assumption that the language $\mathcal{L}$ has a disjunction, the "canonical" extension of a supraclassical, left-absorbing, right-absorbing, deductive and cumulative finitary operation $\mathcal{F}$ is the $\mathcal{C}_{\mathcal{F}}$ described in Corollary 3, and is supraclassical, left-absorbing, right-absorbing, deductive and cumulative. This operation $\mathcal{C}_{\mathcal{F}}$ is, trivially, compact and co-compact.

6 Conclusion, open problems and acknowledgments

In our introduction we mentioned a number of nonmonotonic systems. We may now try to tell which ones have antitonic representations. As mentioned in the introduction, the Closed World Assumption of [10] is explicitly presented by an antitonic representation. Minker's [8] Generalized Closed world Assumption, on the contrary, does not have any such representation since the inference operation it defines does not satisfy Deductivity. Indeed from the assumptions $\{p, p \vee q\}$, GCWA will conclude $\neg q$, but from $p \vee q$ it will not conclude $p \rightarrow \neg q$. Notice that, equivalently, GCWA does not satisfy Or since it will conclude $\neg q$ from p and conclude $\neg p$ from q, but will not conclude $\neg p \vee \neg q$ from $p \vee q$.

Default Logic [11] is a bit more problematic to study since it does not explicitly defines an inference operation. If we take the reasonable, skeptical approach to this definition, we see that, even when only normal defaults are considered, it lacks an antitonic representation, since its inference operation is not deductive (or does not satisfy Or). The example given above for GCWA translates immediately in normal default logic. It is all the more remarkable that, if we restrict ourselves to finite sets of normal defaults without prerequisites, the skeptical inference operation defined is deductive and admits an antitonic representation. Indeed, normal defaults without prerequisites are equivalent to Poole systems without constraints. In [3], Theorems 7.17 and previous theorems, it was shown that, if the language $\mathcal{L}$ has a contradiction (i.e. if any inconsistent set has a finite subset that is inconsistent), the inference operation defined by any finite Poole system without constraints satisfies the conditions of Theorem 4, part 3. For Circumscription, since the inference operation defined by Circumscription may be defined by some preferential model, it is clearly supraclassical, right-absorbing, left-absorbing and deductive. When the so-called *well-foundedness* assumption is satisfied, it is also cumulative. If the language is logically finite, it is also, obviously, compact and therefore has an antitonic representation by Theorem 4. We do not know yet whether it is compact even when the language is infinite. It follows from [6] Section 5.8 that, when the knowledge base is admissible, rational closure has an antitonic representation.

We want to thank Michael Freund for helping us to show that cumulative compact operations are supracompact and David Makinson, Michael Freund and three anonymous referees for their remarks on a draft of this paper.

References

[1] Stefan Brass. On the semantics of supernormal defaults. In Ruzena Bajcsy, editor, *Proceedings of the 13th I.J.C.A.I.*, pages 578–583. Morgan Kaufmann, Chambéry, Savoie, France, August 1993.

[2] Michael Freund. Supracompact inference operations. *Studia Logica*, 52:457–481, 1993.

[3] Michael Freund and Daniel Lehmann. Nonmonotonic inference operations. *Bulletin of the IGPL*, 1(1):23–68, July 1993. Produced by the Max-Planck-Institut für Informatik, Im Stadtwald, D-66123 Saarbrücken, Germany.

[4] Peter Gärdenfors and David Makinson. Nonmonotonic inference based on expectations. *Artificial Intelligence*, 65(1), January 1994.

[5] Sarit Kraus, Daniel Lehmann, and Menachem Magidor. Nonmonotonic reasoning, preferential models and cumulative logics. *Artificial Intelligence*, 44(1–2):167–207, July 1990.

[6] Daniel Lehmann and Menachem Magidor. What does a conditional knowledge base entail? *Artificial Intelligence*, 55(1):1–60, May 1992.

[7] John McCarthy. Circumscription, a form of non monotonic reasoning. *Artificial Intelligence*, 13:27–39, 1980.

[8] Jack Minker. On indefinite databases and the closed world assumption. In D.W. Loveland, editor, *6th Conference on Automated Deduction*, volume 138 of *Lecture Notes in Computer Science*, pages 292–308. Springer-Verlag, New York, June 1982.

[9] David Poole. A logical framework for default reasoning. *Artificial Intelligence*, 36:27–47, 1988.

[10] Raymond Reiter. On closed world data bases. In H. Gallaire and J. Minker, editors, *Logic and Data Bases*, pages 55–76. Plenum, New York / London, 1978.

[11] Raymond Reiter. A logic for default reasoning. *Artificial Intelligence*, 13:81–132, 1980.