Harmonic Analysis of Boolean Functions, and applications in CS

Lecture 6

April 7, 2008

Lecturer: Guy Kindler | Scribe by: Sergey Novikov | Updated: June 2, 2008

In this lecture we show Condorcet Paradox , de¯ne voting schemes , vting schemes properties , and start Kalai's proof of Arrow's Theorem.

1 Condorcet and Condorcet Paradox

1.1 Condorcet

Marie Jean Antoine Nicolas de Caritat, marquis de Condorcet (September 17, 1743 March 28, 1794) was a French philosopher, mathematician, and early political scientist who devised the concept of a Condorcet method. Unlike many of his contemporaries, he advocated a liberal economy, free and equal public education, constitutionalism, and equal rights for women and people of all races. His ideas and writings were said to embody the ideals of the Age of Enlightenment and rationalism, and remain in°uential to this day. He died a mysterious death in prison after a period of being a fugitive from French Revolutionary authorities.

Taken from wikipedia.

1.2 Condorcet Paradox

The Condorcet paradox is a situation in which collective preferences can be cyclic (not rational (see below for de¯nition)), even if the preferences of every individual voter are rationales. This is paradoxical, because it means that majority wishes can be in con°ict with each other.
Example of the paradox:
Suppose we have 3 candidates a,b,c and 3 voters with their preferences

Table 1: Condorcet paradox : voters preferences

Order	Voter1	Voter2	Voter3
1	a	b	c
2	b	c	a
3	c	a	b

This mean that voter1 prefer "a" to "b" and "b" to "c".

We say that candidate x better then candidate y i® majority of voters prefer x to y (we will denote it as x & y). In our example we get that:
a & b since voters 1 and 2 prefer a to b.
b & c since voters 1 and 3 prefer b to c.
c & a since voters 2 and 3 prefer c to a.
Thus we get a & b & c & a.

2 Voting Schemes

In this section we will de¯ne voting schemes , they properties and give examples for voting schemes.

Let C be set of candidates.
We need to choose one of them or rank all candidates.
We have n - votes and 8i 2 $[n]$, R_i-permutation over C.
A social choice function is function $F(R_1, R_2, \ldots, R_n)$ that outputs asymmetric relation R on C. (Asymmetric relation - 8a, b 2 C either aRb or bRa , but not both.)

2.1 Properties of Voting scheme (They may or not have)

1. **R**ationality: R is rational if it is order function.
   It means that if "a" preferred to "b" and "b" preferred to "c" then "a" preferred to "c". Social choice function F is rational if $F(R_1, R_2, \ldots, R_n)$ is always rational.

2. **I**ndependent of irrelevant choices: if 8a, b 2 C , $aF(R_1, R_2, \ldots, R_n)b$ can be determined by looking only on fi : aR_ibg.

3. **N**eutrality: if F is independent of permutations on C.

4. **T**ransitive: if F is invariant under some transitive group of permutations over the voters. (This mean that all voter are equal in some sense)

2.2 Examples:

Majority (prefer "a" to "b" iff most voters prefer "a" to "b") (Assuming number of voters is odd)

- ² Rationality - No (See Condorcet Paradox).
- ² Independent of irrelevant choices - Yes
- ² Neutrality - Yes
- ² Transitive - Yes (in fact it is completely symmetric)

Dictatorship One voter decide everything

- ² Rationality - yes since we look only on preferences pi329(pre3.45-13.533(v)27r639(v)27r639(

- Neutrality - yes
- Transitive - no

Condorcet Method (Was not defined in class)

- Rationality - yes
- Independent of irrelevant choices - no
- Neutrality - yes
- Transitive - yes

Question: Can we have all 4 properties ?

Theorem 1 *Arrow 1950.*
Can't get all properties in one voting scheme.
Or can be phrased as "Dictatorship is the only rational social choice function."

Theorem 2 *Kalai 2001.*
$\exists$ *const* $c < 1$ *s.t. if* F *has properties : independent of irrelevant choices, neutrality, transitive then* $Pr_{R_1 R_2,\dots,R_n}[F(R_1, R_2, \dots, R_n)$ *is rational* $] \leq c < 1$.

Statement: $\exists$ const k s.t. if F has property : independent of irrelevant choices, neutrality and $\epsilon = Pr[F$ is irrational$]$ then F is a most $k \cdot \epsilon$-far from dictatorship.

Proof In proof we will look on case of 3 candidates (if we have more then 3 candidates then by the property of "independent of irrelevant choices" we can consider only on 3 of them independently on all others).
Let x_i, y_i, z_i $i = 1, \dots, n$ determine if i-th voter prefers "a" to "b" ,"b" to "c" or "c" to "a" respectively. (If $x_7 = 1$ then 7-th voter prefer "a" to "b" , and if $x_7 = -1$ then 7-th voter prefer "b" to "a").
Note that attention that if $\begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ it means that "a" preferred to "b" , "b" to "c" and "c" to "a" which is illegal since F is rational. By same argument $\begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$ is illegal as well.
If F "independent of irrelevant choices" then $F(x, y, z) = (f(x), g(y), h(z))$.
Where $f, g, h : \{\pm 1\}^n \to \{\pm 1\}$ are functions representing the preferences of the voters on 2 of 3 candidates (i.e. f gets for each voter if "a" preferred to "b" and outputs if F prefers "a" to "b")
F is rational at (x, y, z) iff $\begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} \notin \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \right\}$
Denote: $NAE(\alpha, \beta, \gamma) = 1_{\left\{ \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \notin \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \right\} \right\}}$
Lets write the $NAE(\alpha, \beta, \gamma)$ as multi-linear polynomial of α, β, γ.

$\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$ 2/ $\left\{ \begin{pmatrix}1\\1\\1\end{pmatrix} \begin{pmatrix}¡1\\¡1\\¡1\end{pmatrix} \right\}$) $(\alpha+\beta+\gamma)$ 2 f1, ¡1g) $(\alpha+\beta+\gamma)^2 = 1$
) $NAE(\alpha,\beta,\gamma)$ = ¡ $\frac{(\alpha+\beta+\gamma)^2-9}{8}$ = ¡ $\frac{\alpha^2+\beta^2+\gamma^2+2\alpha\beta+2\beta\gamma+2\alpha\gamma-9}{8}$ = ¡ $\frac{3+2\alpha\beta+2\beta\gamma+2\alpha\gamma-9}{8}$ = $\frac{3}{4}$ ¡ $\frac{1}{4}\alpha\beta$ ¡ $\frac{1}{4}\beta\gamma$ ¡ $\frac{1}{4}\alpha\gamma$
De¯ne ª = f(x,y,z)j8i $NAE(x_i,y_i,z_i) = 1$g and $A(x,y,z) = 1_\Psi$.
Then we can write:
$\Pr_{R_1R_2,\ldots,R_n}[F$ is rational$] = \Pr_{R_1R_2,\ldots,R_n}[NAE(f(x),g(y),h(z)) = 1] =$
$E_\Psi[NAE(f(x),g(y),h(z))] = \frac{1}{\Pr_{x,y,z}[\Psi]} E_{(x,y,z)\in\{\pm1\}^{3n}}[A(x,y,z)NAE(f(x),g(y),h(z))] =$
$\frac{1}{\Pr_{x,y,z}[\Psi]} < A(x,y,z), NAE(f(x),g(y),h(z)) >$

Now Lets calculate every term separately .
$\Pr_{x,y,z}$[ª] = $(\frac{3}{4})^n$
$NAE(f(x),g(y),h(z)) = \frac{3}{4}$ ¡ $\frac{1}{4}f(x)g(y)\frac{1}{4}f(x)h(z)\frac{1}{4}g(y)h(z)$
$f(x)g(y) = (\sum_{S\subset[n]}\hat{f}(S)\chi_s(x))\sum_{T\subset[n]}\hat{g}(T)\chi_T(y) =$
$\sum_{S\subset[n]T\subset[n]}\hat{f}(S)\hat{g}(T)\chi_S(x)\chi_T(y)$ The proof will be continued ■

3 Appendix: Condorcet (voting) method

Taken from wiki and $http : //minguo.info/election_methods/condorcet/condorcet_voting_explained$

3.1 Short description

² Rank the candidates in order (1st, 2nd, 3rd, etc.) of preference. Tie rankings are allowed, which express no preference between the tied candidates.

² Comparing each candidate on the ballot to every other, one at a time (pairwise), tally a "win" for the victor in each match.

² Sum these wins for all ballots cast. The candidate who has won every one of their pairwise contests is the most preferred, and hence the winner of the election.

² In the event of a tie, use a resolution method.

A particular point of interest is that it is possible for a candidate to be the most preferred overall without being the ¯rst preference of any voter. In a sense, the Condorcet method yields the "best compromise" candidate, the one that the largest majority will ¯nd to be least disagreeable, even if not their favorite.