Harmonic Analysis of Boolean Functions, and applications in CS

Lecture 14

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1 In Previous Lectures

In the previous talk we proved Freidgoods theorem :

Theorem 1 (F'95) *If $I(f) \leq k$ then f is (";$\frac{C^{k/\varepsilon}}{\varepsilon^3}$)*

We also saw that this result is not so robust, because if we look at an altered function, (a function that its values were altered with some probability) then the influence jumps to values that the theorem would be meaningless to them.

Claim 2 *If $I(f) \leq k$, $f'(x) = \begin{cases} f(x) & 1-\alpha \\ -f(x) & \alpha \end{cases}$ then $I(f') \geq$ (n):*

there are robust analouges of Theorem 1, for example, in a paper titeled "tails of fourier spectrum of boolean functions" Bourgain proves such an anlouge which is a generalization of FKN to the k-th degree. We also proved the KKL theorem,

Theorem 3 (KKL 88') *If $I(f) \leq \frac{1-\hat{f}(\emptyset)^2}{20} log(\frac{1}{\delta})$ then $\exists i$ $I'_i(f) \leq C(1-t^2)\frac{log(cn)}{n}$.*

We also defined the analougue influence for the cube,

Definition 4 *Let $f : \{\pm 1\}^n \to \{\pm 1\}$. $I'_i(f) = \Pr_{x \setminus i}$ (f(x) is non constant on x_i)*

$$I'(f) = \sum_i I'_i(f) \tag{1}$$

After we have a definition of influence on the cube, we also introduced a version of KKL to the cube

Theorem 5 (BKKKL) *Let $f : [0;1]^n \to \{\pm 1\}$ and let $t := E(f)$. Then $\exists i$ s.t $I'_i(f) \leq C(1-t^2)\frac{log(cn)}{n}$*

2 Graph properties and boolean functions

In this lecture we will try to apply our theory of boolean functions into graph properties, we will define graph properties in boolean functions language and then we will immediatly get some interesting results on graph properties. Lets begin with a usefull definition,

Definition 8 f *is monotone if* $(x \le y) \to (f(x) \le f(y))$

Define graph property in boolean functions language

Definition 9 f *is a graph property if :*

1. *$\exists$ 1-1 association between coordinates and edges of a complete graph.*
2. *f is invariant under vertex permutations(so f is transitive, but isn't necesserily simetric).*

Now let's define some usefull functions, φ and ψ

Definition 10

$$\varphi_f(\overline{p}) = \Pr_{x \sim \mu_{\overline{p}}^{(n)}} (f(x) = -1)$$

$$\psi_f(p) = \varphi_f(p, \ldots, p)$$

2.1 Derivatives of φ and ψ

we now compute the derivatives of φ *and* ψ*:to*

$$\varphi_f(\overline{p}) = \sum_{x \in \{\pm 1\}^n} \mu_{\overline{p}}(x) \mathbb{1}_{\{f(x)=-1\}} =$$

$$= \sum_{x_1, \ldots, x_{n-1}} \mu^{(n-1)}_{(p_1, \ldots, p_{n-1})}(x_1, \ldots, x_{n-1}) \left[p_n \mathbb{1}_{\{f(x_1, \ldots, x_{n-1}, -1) = -1\}} + (1 - p_n) \mathbb{1}_{\{f(x_1, \ldots, x_{n-1}, 1) = 1\}}\right]$$

(2)

So we have:

$$\frac{\partial \varphi_p}{\partial p_n}(\overline{p}) = \sum_{x_1, \ldots, x_{n-1}} \mu^{(n-1)}_{(p_1, \ldots, p_{n-1})}(x_1, \ldots, x_{n-1}) \mathbb{1}_{\{f(x_1, \ldots, x_{n-1}, \cdot) \text{ is not-const.}\}} = I_n^{\overline{p}}(f)$$

and similarily:

$$\forall i \quad \frac{\partial \varphi_p}{\partial p_i}(\overline{p}) = I_i^{\overline{p}}(f).$$

We have (Denoting $\gamma(p) = (p, \ldots, p)$)

$$(\psi_f(p))' = (\varphi_f \circ \gamma(p))' = \left\langle \nabla \varphi_f(\gamma(p)), \gamma'(p) \right\rangle =$$

$$= \left\langle (I_1^{(p, \ldots, p)}(f), \ldots, I_n^{(p, \ldots, p)}(f)), (1, 1, \ldots, 1) \right\rangle = \sum_i I_i^p(f) = I^p(f)$$

2.2 Phase transition in graph properties

Now, we will prove a result about phase changes in graph properties. Let's assume we have a graph property P, such as: "the graph G has a triangle". If we consider this graph property over graphs of size 100 with a 0.012 probability of having an edge, then the probability for having a triangle is about 0.25. But if we look at the same property P over graphs of the same size but with a probability of 0.02 for having each edge, than the probability of having an edge is about 0.75. In fact this property exists for all graph properties, there is a very swift and short change in the probablity of having an edge between the phases "the property will never happen" and "the property will happen always"

Theorem 11 (FK) *Let f be a monotone graph property , and let $\ _f$ be as in definition 8. Then $\ _f(r) \geq ''$ implies $\ _f(r + \frac{\log(\frac{1}{2\varepsilon})}{c\log(n)}) \geq 1 - ''$.*

Proof

From [BKKKL] we have :

$$\forall p\ \exists i\ I_i^p(f) \geq C'(1 - (\mathbb{E}_{\mu_p}(f))^2)\frac{\log(n)}{n}$$

.

But f is transitive, so $I^p(f) \geq C'(1 - (\mathbb{E}_{\mu_p}(f))^2)\log(n)$.

Now, let's express the expectancy in the means of $\ _f(p)$,

$$\mathbb{E}_{\mu_p}(f) = \Pr_{\mu_p}(f = 1) - \Pr_{\mu_p}(f = -1) = 1 - \ _f(p) - \ _f(p) = 1 - 2\ _f(p)$$

. This implies

$$(\ _f(p))' = I^p(f) \geq C'4\ _f(p)(1 - \ _f(p))\log(n) = C''\ _f(p)(1 - \ _f(p))\log(n)$$

(where C'' is a constant). Suppose $\ _f(p) \leq \frac{1}{2}$. Then from the inequality above we get: $(\ _f(p))' \geq \frac{C''}{2}\ _f(p)\log(n)$.

$$(\log(\ _f(p)))' = \frac{(\ _f(p))'}{\ _f(p)} \geq C\log(n)$$

We know that the derivative of $\log(\ _f(p))$ is higher than $C\log(n)$, so

$$\log(\ _f(r + \frac{\log(\frac{1}{2\varepsilon})}{c\log(n)})) \geq \log(\ _f(r)) + \log(\frac{1}{2}) - \log('') \geq \log(\frac{1}{2})$$

From the monotonicity of the log function we know $a \geq b$ then $\log(a) \geq \log(b)$. and therefore

$$\ _f(r + \frac{log(\frac{1}{2\varepsilon})}{c\log(n)}) \geq \frac{1}{2}$$

. If we repeat the same argument on the negative property $\tilde{f}$ ($\tilde{f}(G) = -f(G)$) we get that $\ _{\tilde{f}}(r) \leq ('')$ implies $\ _{\tilde{f}}(r + \frac{\log(\frac{1}{2\varepsilon})}{c\log(n)}) \geq \frac{1}{2}$. We can translate it to $\ _f(r) \geq (1 - '')$ implies

$_f(r + \frac{\log(\frac{1}{2\varepsilon})}{c\log(n)}) \leq \frac{1}{2}$. f2cm77f So by combining the two arguments we get