Harmonic Analysis of Boolean Functions in Computer Science

Assignment no. 2

Date due: 3/31/2007

1. Let $f : \{-1,1\}^n \to \mathbb{R}$ be a function, and define $\tau(p) \doteq \|f\|_p$ for all $p \geq 1$.
   (a) Show that τ is monotone increasing.
   Hint: recall that the function $t \to |t|^q$ is convex for every $q \geq 1$ and use Jensen's inequality, which states that for any convex function φ and any random variable X, $\mathbb{E}[\varphi(x)] \geq \varphi(\mathbb{E}[X])$.
   (b) Show that $\lim_{p\to\infty} \varphi(p) = \|f\|_\infty$.
2. For a function $f : \{-1,1\}^n \to \mathbb{R}$, let $\sigma(f) \doteq \sum_i \widehat{f}(i)$.
   (a) Denote $w(x) \doteq \sum x_i$, and show that $\sigma(f) = \mathbb{E}[f(x)w(x)]$ for every f.
   (b) Prove that among all Boolean-valued functions over $\{-1,1\}^n$, σ is maximized by the Majority function.
   (c) Deduce that majority has the largest total influence of all Boolean-valued monotone functions.
3. Let $f : \{-1,1\} \to \mathbb{R}$. For a set $T \subseteq [n]$ of coordinates we define the variation of f on T by
$$\mathsf{Vr}_T(f) \doteq \mathop{\mathbb{E}}_{x\setminus T}\left[\mathop{\mathbb{V}}_{x\cap T}[f(x)]\right].$$
   (a) Write $\mathsf{Vr}_T(f)$ in terms of the Fourier coefficients of f.
   (b) Let $g : \{-1,1\} \to \mathbb{R}$ be a function that depends only on coordinates from T. Prove that
$$\|f-g\|_2^2 \geq \mathsf{Vr}_{[n]\setminus T}(f).$$
   (c) Prove that if $\mathsf{Vr}_{[n]\setminus T}(f) \leq \epsilon$ then there exists a function $g : \{-1,1\} \to \mathbb{R}$ that depends only on coordinates from T, such that
$$\|f-g\|_2^2 \leq \mathsf{Vr}_{[n]\setminus T}(f).$$
4. Let $g : \{-1,1\}^n \to \mathbb{R}$ be a j-junta, and let $f : \{-1,1\}^n \to \mathbb{R}$ be a symmetric function (namely, for any permutation π of coordinates and every x, $f(x) = f(\pi(x))$). Show that
$$\|f-g\|_2^2 \geq \frac{n-j}{n} \cdot \mathbb{V}[f].$$
   (For extra credit, prove the same when f is only known to be transitive.)
5. (a) write $\mathbb{E}_{x,y}[f(x)g(y)h(xy)]$ in terms of the Fourier coefficients of f, g, and h.
   (b) Assume that $\|f\|_2 = 1$ and that $\mathbb{E}_{x,y}[f(x)f(y)f(xy)] \geq 1-\epsilon$. Prove that $\widehat{f}(S) \geq (1-\epsilon)$ for some $S \subseteq [n]$.